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**MAXWELL EQUATIONS IN COMPLEX FORM OF MAJORANA -  
OPPENHEIMER, SOLUTIONS WITH CYLINDRIC SYMMETRY  
IN RIEMANN  $S_3$  AND LOBACHEVSKY  $H_3$  SPACES**

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Complex formalism of Riemann - Silberstein - Majorana - Oppenheimer in Maxwell electrodynamics is extended to the case of arbitrary pseudo-Riemannian space - time in accordance with the tetrad recipe of Tetrode - Weyl - Fock - Ivanenko. In this approach, the Maxwell equations are solved exactly on the background of static cosmological Einstein model, parameterized by special cylindrical coordinates and realized as a Riemann space of constant positive curvature. A discrete frequency spectrum for electromagnetic modes depending on the curvature radius of space and three parameters is found, and corresponding basis electromagnetic solutions have been constructed explicitly. In the case of elliptical model a part of the constructed solutions should be rejected by continuity considerations.

Similar treatment is given for Maxwell equations in hyperbolic Lobachevsky model, the complete basis of electromagnetic solutions in corresponding cylindrical coordinates has been constructed as well, no quantization of frequencies of electromagnetic modes arises.

## 1 Introduction

It is well-known that Special Relativity arose from investigation of symmetry properties of the Maxwell equations with respect to inertial motion of the reference frame: Lorentz [1], Poincaré [2], Einstein [3]. Naturally, it was electromagnetic field that was the first and principal object for the Special Relativity: Minkowski [4], Silberstein [5], Marcolongo [7], Bateman [8]. In 1931 Majorana [10] and Oppenheimer [9] proposed to consider classical Maxwell equations as a quantum photon equations. In this context they introduced 3-vector function obeying Dirac-like massless wave equation. It turned out that much earlier in 1907 the same mathematical translation of classical Maxwell theory was performed by Silberstein [5]; besides, he noted himself that the same approach was used earlier by Riemann [6]. That history was much forgotten, and many years this complex approach to electrodynamics was connected mainly with Majorana and Oppenheimer. Historical justice was rendered by Bialynicki-Birula [11], see also in [12-17].

In the present paper <sup>1</sup> we use the complex formalism of Riemann – Silberstein – Majorana – Oppenheimer in Maxwell electrodynamics extended to the case of arbitrary pseudo-Riemannian space – time in accordance with the tetrad recipe of Tetrode – Weyl – Fock – Ivanenko (for more detail, see [19]). In this approach the Maxwell equations are solved exactly on the background of simplest static cosmological models, Riemann and Lobachevsky spaces of constant

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<sup>1</sup>It is an extended version of the paper: Bogush A.A., Krylov G.G., Ovsyuk E.M., Red'kov V.M., Maxwell electrodynamics in complex form, solutions with cylindric symmetry in Riemann space of constant positive curvature. Doklady of the National Academy of Sciences of Belarus. 2009 (in press).

curvature parameterized by cylindric coordinate (many years ago these coordinates were used by Schrödinger in his book [19]; systematic treatment of coordinate systems in Riemann and Lobachevsky spaces was given by Olevsky [21]). In the case of compact Riemann model a discrete frequency spectrum for electromagnetic modes depending on the curvature radius of space is found. In the case of hyperbolic Lobachevsky model no discrete spectrum for frequencies of electromagnetic modes arises.

## 2 Complex matrix form of Maxwell equations

Let us start with the Maxwell equations in vacuum:

$$\begin{aligned} \operatorname{div} c\mathbf{B} &= 0, & \operatorname{rot} \mathbf{E} &= -\frac{\partial c\mathbf{B}}{\partial ct}, \\ \operatorname{div} \mathbf{E} &= \frac{\rho}{\epsilon_0}, & \operatorname{rot} c\mathbf{B} &= \mu_0 c\mathbf{J} + \frac{\partial \mathbf{E}}{\partial ct}. \end{aligned} \quad (2.1)$$

With notation  $j^a = (\rho, \mathbf{J}/c)$ ,  $c^2 = 1/\epsilon_0\mu_0$  they read

$$\begin{aligned} \operatorname{div} c\mathbf{B} &= 0, & \operatorname{rot} \mathbf{E} &= -\frac{\partial c\mathbf{B}}{\partial ct}, \\ \operatorname{div} \mathbf{E} &= \frac{\rho}{\epsilon_0}, & \operatorname{rot} c\mathbf{B} &= \frac{\mathbf{j}}{\epsilon_0} + \frac{\partial \mathbf{E}}{\partial ct}. \end{aligned} \quad (2.2)$$

or in the explicit component form

$$\begin{aligned} \partial_1 cB^1 + \partial_2 cB^2 + \partial_3 cB^3 &= 0, & \partial_2 E^3 - \partial_3 E^2 + \partial_0 cB^1 &= 0, \\ \partial_3 E^1 - \partial_1 E^3 + \partial_0 cB^2 &= 0, & \partial_1 E^2 - \partial_2 E^1 + \partial_0 cB^3 &= 0, \\ \partial_1 E^1 + \partial_2 E^2 + \partial_3 E^3 &= j^0/\epsilon_0, & \partial_2 cB^3 - \partial_3 cB^2 - \partial_0 E^1 &= j^1/\epsilon_0, \\ \partial_3 cB^1 - \partial_1 cB^3 - \partial_0 E^2 &= j^2/\epsilon_0, & \partial_1 cB^2 - \partial_2 cB^1 - \partial_0 E^3 &= j^3/\epsilon_0. \end{aligned} \quad (2.3)$$

With the use of complex 3-vector field  $\psi^k = E^k + icB^k$  eqs. (2.3) can be combined into

$$\begin{aligned} \partial_1 \Psi^1 + \partial_2 \Psi^0 + \partial_3 \Psi^3 &= j^0/\epsilon_0, \\ -i\partial_0 \psi^1 + (\partial_2 \psi^3 - \partial_3 \psi^2) &= i j^1/\epsilon_0, \\ -i\partial_0 \psi^2 + (\partial_3 \psi^1 - \partial_1 \psi^3) &= i j^2/\epsilon_0, \\ -i\partial_0 \psi^3 + (\partial_1 \psi^2 - \partial_2 \psi^1) &= i j^3/\epsilon_0. \end{aligned} \quad (2.4)$$

These four equations can be presented in the matrix form:

$$\left[ \begin{array}{c} -i\partial_0 \\ \partial_1 \\ \partial_2 \\ \partial_3 \end{array} \begin{vmatrix} a_0 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 \\ a_2 & 0 & 1 & 0 \\ a_3 & 0 & 0 & 1 \end{vmatrix} + \begin{array}{c} \partial_1 \\ \partial_2 \\ \partial_3 \end{array} \begin{vmatrix} b_0 & 1 & 0 & 0 \\ b_1 & 0 & 0 & 0 \\ b_2 & 0 & 0 & -1 \\ b_3 & 0 & 1 & 0 \end{vmatrix} + \begin{array}{c} \partial_2 \\ \partial_3 \end{array} \begin{vmatrix} c_0 & 0 & 1 & 0 \\ c_1 & 0 & 0 & 1 \\ c_2 & 0 & 0 & 0 \\ c_3 & -1 & 0 & 0 \end{vmatrix} + \begin{array}{c} \partial_3 \end{array} \begin{vmatrix} d_0 & 0 & 0 & 1 \\ d_1 & 0 & -1 & 0 \\ d_2 & 1 & 0 & 0 \\ d_3 & 0 & 0 & 0 \end{vmatrix} \right] \begin{vmatrix} 0 \\ \psi^1 \\ \psi^2 \\ \psi^3 \end{vmatrix} = \frac{1}{\epsilon_0} \begin{vmatrix} j^0 \\ i j^1 \\ i j^2 \\ i j^3 \end{vmatrix}.$$

There arise four matrices (including arbitrary numerical parameters)

$$(-i\alpha^0\partial_0 + \alpha^j\partial_j)\Psi = J, \quad \Psi = \begin{vmatrix} 0 \\ \psi^1 \\ \psi^2 \\ \psi^3 \end{vmatrix}, \quad \alpha^0 = \begin{vmatrix} a_0 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 \\ a_2 & 0 & 1 & 0 \\ a_3 & 0 & 0 & 1 \end{vmatrix},$$

$$\alpha^1 = \begin{vmatrix} b_0 & 1 & 0 & 0 \\ b_1 & 0 & 0 & 0 \\ b_2 & 0 & 0 & -1 \\ b_3 & 0 & 1 & 0 \end{vmatrix}, \quad \alpha^2 = \begin{vmatrix} c_0 & 0 & 1 & 0 \\ c_1 & 0 & 0 & 1 \\ c_2 & 0 & 0 & 0 \\ c_3 & -1 & 0 & 0 \end{vmatrix}, \quad \alpha^3 = \begin{vmatrix} d_0 & 0 & 0 & 1 \\ d_1 & 0 & -1 & 0 \\ d_2 & 1 & 0 & 0 \\ d_3 & 0 & 0 & 0 \end{vmatrix} \quad (2.5)$$

and

$$(\alpha^0)^2 = \begin{vmatrix} a_0a_0 & 0 & 0 & 0 \\ a_1a_0 + a_1 & 1 & 0 & 0 \\ a_2a_0 + a_2 & 0 & 1 & 0 \\ a_3a_0 + a_3 & 0 & 0 & 1 \end{vmatrix}.$$

Let us require

$$(\alpha^0)^2 = +I, \quad a_0a_0 = 1, \quad a_1a_0 + a_1, \quad a_2a_0 + a_2, \quad a_3a_0 + a_3;$$

the most simple solution is

$$a_0 = \pm 1, \quad a_j = 0, \quad \alpha^0 = \begin{vmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}, \quad (\alpha^0)^2 = +I. \quad (2.6)$$

In the same manner

$$(\alpha^1)^2 = \begin{vmatrix} b_0^2 + b_1 & b_0 & 0 & 0 \\ b_1b_0 & b_1 & 0 & 0 \\ b_2b_0 - b_3 & b_2 & -1 & 0 \\ b_3b_0 - b_2 & b_3 & 0 & -1 \end{vmatrix}, \quad (\alpha^1)^2 = -I,$$

we get

$$b_0 = 0, \quad b_1 = -1, \quad b_2 = 0, \quad b_3 = 0, \quad \alpha^1 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix}. \quad (2.7)$$

Analogously

$$(\alpha^2)^2 = \begin{vmatrix} c_0c_0 + c_2 & 0 & c_0 & 0 \\ c_1c_0 + c_3 & -1 & c_1 & 0 \\ c_2c_0 & 0 & c_2 & 0 \\ c_3c_0 - c_1 & 0 & c_3 & -1 \end{vmatrix} = -I,$$

that is

$$c_0 = 0, \quad c_1 = 0, \quad c_2 = -1, \quad c_3 = 0, \quad \alpha^2 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}, \quad (\alpha^2)^2 = -I. \quad (2.8)$$

And finally

$$(\alpha^3)^2 = \begin{vmatrix} d_0 d_0 + d_3 & 0 & 0 & d_0 \\ d_1 d_0 - d_2 & -1 & 0 & 0 \\ d_2 d_0 + d_1 & 0 & -1 & d_2 \\ d_3 d_0 & 0 & 0 & d_3 \end{vmatrix} = -I,$$

that is

$$d_0 = 0, \quad d_1 = 0, \quad d_2 = 0, \quad d_3 = -1, \quad \alpha^3 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix}, \quad (\alpha^3)^2 = -I. \quad (2.9)$$

Consider their products:

$$\alpha^1 \alpha^2 = -\alpha^2 \alpha^1 = +\alpha^3, \quad \alpha^2 \alpha^3 = -\alpha^3 \alpha^2 = \alpha^1, \quad \alpha^3 \alpha^1 = -\alpha^1 \alpha^3 = \alpha^2. \quad (2.10)$$

Consider the product  $\alpha^0 \alpha^i$ :

$$k = \pm 1, \quad \alpha^0 \alpha^1 = \begin{vmatrix} 0 & k & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix}, \quad \alpha^1 \alpha^0 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -k & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix}.$$

If  $k = +1$ , we will have the most simple commutation rule:

$$\alpha^0 = I, \quad \alpha^i \alpha^0 = \alpha^0 \alpha^i = \alpha^i. \quad (2.11)$$

Thus, the Maxwell matrix equation looks

$$(-i\partial_0 + \alpha^j \partial_j) \Psi = J, \quad \Psi = \begin{vmatrix} 0 \\ \psi^1 \\ \psi^2 \\ \psi^3 \end{vmatrix}, \quad J = \frac{1}{\epsilon_0} \begin{vmatrix} j^0 \\ i j^1 \\ i j^2 \\ i j^3 \end{vmatrix},$$

$$\alpha^1 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix}, \quad \alpha^2 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}, \quad \alpha^3 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix},$$

$$(\alpha^1)^2 = -I, \quad (\alpha^2)^2 = -I, \quad (\alpha^3)^2 = -I,$$

$$\alpha^1 \alpha^2 = -\alpha^2 \alpha^1 = \alpha^3, \quad \alpha^2 \alpha^3 = -\alpha^3 \alpha^2 = \alpha^1, \quad \alpha^3 \alpha^1 = -\alpha^1 \alpha^3 = \alpha^2. \quad (2.12)$$

### 3 Maxwell matrix equation in Riemannian space

Maxwell equation

$$(\alpha^0 \partial_0 + \alpha^j \partial_j) \Psi = J, \quad \alpha^0 = -iI, \quad \Psi = \begin{vmatrix} 0 \\ \mathbf{E} + ic\mathbf{B} \end{vmatrix}, \quad J = \frac{1}{\epsilon_0} \begin{vmatrix} \rho \\ i\mathbf{j} \end{vmatrix} \quad (3.1)$$

can be extended to an arbitrary Riemannian space-time in accordance with general tetrad recipe of Tetrode-Weyl-Fock-Ivanenko (see [18]):

$$\alpha^\rho(x) [\partial_\rho + A_\rho(x)] \Psi(x) = J(x), \quad \alpha^\rho(x) = \alpha^c e_{(c)}^\rho(x), \quad A_\rho(x) = \frac{1}{2} j^{ab} e_{(a)}^\beta \nabla_\rho e_{(n)\beta} . \quad (3.2)$$

where  $e_{(c)}^\rho(x)$  is a tetrad;  $j^{ab}$  stands for generators for complex vector representation of orthogonal group  $SO(3.C)$ ;  $\nabla_\rho$  denotes a covariant derivative. Eq. (3.1) can be rewritten differently

$$\alpha^c (e_{(c)}^\rho \partial_\rho + \frac{1}{2} j^{ab} \gamma_{abc}) \Psi = J(x), \quad (3.3)$$

with the use of Ricci rotation coefficients  $\gamma_{bac} = -\gamma_{abc} = -e_{(b)\beta;\alpha} e_{(a)}^\beta e_{(c)}^\alpha$ .

Eq. (3.1) is invariant under gauge transformations of the local Lorentz group (see [18])

$$\begin{aligned} \Psi'(x) &= S(x) \Psi(x), \quad S(x) \in SO(3.C)_{loc}, \\ e'_{(a)\alpha}(x) &= L_a^b(x) e_{(b)\alpha}(x), \\ \alpha^\rho(x) [\partial_\rho + A_\rho(x)] \Psi(x) &= J(x), \\ \alpha'^\rho(x) [\partial_\rho + A'_\rho(x)] \Psi'(x) &= J'(x). \end{aligned} \quad (3.4)$$

### 4 Tetrad explicit form of Maxwell matrix equation

Matrix Maxwell equation (3.3) can be written as

$$-i (e_{(0)}^\rho \partial_\rho + \frac{1}{2} j^{ab} \gamma_{ab0}) \Psi + \alpha^k (e_{(k)}^\rho \partial_\rho + \frac{1}{2} j^{ab} \gamma_{abk}) \Psi = J(x). \quad (4.1)$$

Taking into account the identities

$$\begin{aligned} \frac{1}{2} j^{ab} \gamma_{ab0} &= [s_1(\gamma_{230} + i\gamma_{010}) + s_2(\gamma_{310} + i\gamma_{020}) + s_3(\gamma_{120} + i\gamma_{030})], \\ \frac{1}{2} j^{ab} \gamma_{abk} &= [s_1(\gamma_{23k} + i\gamma_{01k}) + s_2(\gamma_{31k} + i\gamma_{02k}) + s_3(\gamma_{12k} + i\gamma_{03k})] \end{aligned} \quad (4.2)$$

and using notation

$$\begin{aligned} e_{(0)}^\rho \partial_\rho &= \partial_{(0)}, \quad e_{(k)}^\rho \partial_\rho = \partial_{(k)}, \quad a = 0, 1, 2, 3, \\ (\gamma_{01a}, \gamma_{02a}, \gamma_{03a}) &= \mathbf{v}_a, \quad (\gamma_{23a}, \gamma_{31a}, \gamma_{12a}) = \mathbf{p}_a, \end{aligned} \quad (4.3)$$

eq. (4.1) is reduced to

$$-i \left[ \partial_{(0)} + \mathbf{s}(\mathbf{p}_0 + i\mathbf{v}_0) \right] \Psi + \alpha^k \left[ \partial_{(k)} + \mathbf{s}(\mathbf{p}_k + i\mathbf{v}_k) \right] \Psi = J(x) ,$$

or

$$\begin{aligned} & \left( \alpha^k \partial_{(k)} + \mathbf{s}\mathbf{v}_0 + \alpha^k \mathbf{s}\mathbf{p}_k \right) \left| \begin{array}{c} 0 \\ \mathbf{E} + ic\mathbf{B} \end{array} \right| - \\ & -i \left( \partial_{(0)} + \mathbf{s}\mathbf{p}_0 - \alpha^k \mathbf{s}\mathbf{v}_k \right) \left| \begin{array}{c} 0 \\ \mathbf{E} + ic\mathbf{B} \end{array} \right| = \frac{1}{\epsilon_0} \left| \begin{array}{c} \rho \\ i\mathbf{j} \end{array} \right| , \end{aligned} \quad (4.4)$$

where  $s_i$  stands for the generators

$$s_1 = \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array} \right| , \quad s_2 = \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array} \right| , \quad s_3 = \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right| .$$

## 5 Cylindric coordinates and tetrad in spherical space $S_3$

Let us consider the Maxwell equation in the cylindric coordinates and tetrad in spherical space  $S_3$ :

$$\begin{aligned} n_1 &= \sin r \cos \phi , \quad n_2 = \sin r \sin \phi , \quad n_3 = \cos r \sin z , \quad n_4 = \cos r \cos z ; \\ dS^2 &= dt^2 - dr^2 - \sin^2 r d\phi^2 - \cos^2 r dz^2 , \quad x^\alpha = (t, r, \phi, z) , \end{aligned}$$

$$e_{(a)}^\beta(y) = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sin^{-1} r & 0 \\ 0 & 0 & 0 & \cos^{-1} r \end{array} \right| , \quad e_{(a)\beta}(y) = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -\sin r & 0 \\ 0 & 0 & 0 & -\cos r \end{array} \right| ; \quad (5.1)$$

where  $(r, \phi, z)$  run within

$$\rho \in [0, +\pi/2] , \quad \phi \in [-\pi, +\pi] , \quad z \in [-\pi, +\pi] .$$

Christoffel symbols are given by  $\Gamma_{\beta\sigma}^0 = 0$  ,  $\Gamma_{00}^i = 0$  ,  $\Gamma_{0j}^i = 0$  and

$$\begin{aligned} \Gamma_{jk}^r &= \left| \begin{array}{ccc} 0 & 0 & 0 \\ 0 & -\sin r \cos r & 0 \\ 0 & 0 & \sin r \cos r \end{array} \right| , \\ \Gamma_{jk}^\phi &= \left| \begin{array}{ccc} 0 & \frac{\cos r}{\sin r} & 0 \\ \frac{\cos r}{\sin r} & 0 & 0 \\ 0 & 0 & 0 \end{array} \right| , \quad \Gamma_{jk}^z = \left| \begin{array}{ccc} 0 & 0 & -\frac{\sin r}{\cos r} \\ 0 & 0 & 0 \\ -\frac{\sin r}{\cos r} & 0 & 0 \end{array} \right| . \end{aligned} \quad (5.2)$$

For covariant derivatives of tetrad vectors we get

$$A_{\beta;\alpha} = \frac{\partial A_\beta}{\partial x^\alpha} - \Gamma_{\alpha\beta}^\sigma A_\sigma \quad \implies \quad e_{(0)\beta;\alpha} = \frac{\partial e_{(0)\beta}}{\partial x^\alpha} - \Gamma_{\alpha\beta}^0 e_{(0)0} = 0 ,$$

$$e_{(1)\beta;\alpha} = \Gamma_{\alpha\beta}^r \quad \Longrightarrow \quad e_{(1)\beta;\alpha} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\sin r \cos r & 0 \\ 0 & 0 & 0 & \sin r \cos r \end{vmatrix},$$

$$e_{(2)\beta;\alpha} = \frac{\partial e_{(2)\beta}}{\partial x^\alpha} - \Gamma_{\alpha\beta}^\phi e_{(2)\phi} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \cos r & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix},$$

$$e_{(3)\beta;\alpha} = \frac{\partial e_{(3)\beta}}{\partial x^\alpha} - \Gamma_{\alpha\beta}^z e_{(3)z} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sin r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}.$$

It remains to find Ricci rotation coefficients:

$$\begin{aligned} \gamma_{ab0} &= e_{(a)}^\beta e_{(b)\beta;t} e_{(0)}^t = 0, & \gamma_{ab1} &= e_{(a)}^a e_{(b)a;r} e_{(1)}^r, \\ \gamma_{ab2} &= e_{(a)}^\beta e_{(b)\beta;\phi} e_{(2)}^\phi, & \gamma_{ab3} &= e_{(a)}^\beta e_{(b)\beta;z} e_{(3)}^z; \end{aligned}$$

from which it follow

$$\gamma_{011} = \gamma_{021} = \gamma_{031} = 0, \quad \gamma_{012} = \gamma_{022} = \gamma_{032} = 0, \quad \gamma_{013} = \gamma_{023} = \gamma_{033} = 0,$$

$$\begin{aligned} \gamma_{231} &= 0, & \gamma_{311} &= 0, & \gamma_{121} &= 0, \\ \gamma_{232} &= 0, & \gamma_{312} &= 0, & \gamma_{122} &= \frac{\cos r}{\sin r}, \\ \gamma_{233} &= 0, & \gamma_{313} &= \frac{\sin r}{\cos r}, & \gamma_{123} &= 0. \end{aligned}$$

Taking into account the identities

$$\begin{aligned} e_{(0)}^\rho \partial_\rho &= \partial_{(0)} = \partial_t, & e_{(1)}^\rho \partial_\rho &= \partial_{(1)} = \partial_r, \\ e_{(2)}^\rho \partial_\rho &= \partial_{(2)} = \frac{1}{\sin r} \partial_\phi, & e_{(3)}^\rho \partial_\rho &= \partial_{(3)} = \frac{1}{\cos r} \partial_z, \\ \mathbf{v}_0 &= (\gamma_{010}, \gamma_{020}, \gamma_{030}) \equiv 0, & \mathbf{v}_1 &= (\gamma_{011}, \gamma_{021}, \gamma_{031}) \equiv 0, \\ \mathbf{v}_2 &= (\gamma_{0120}, \gamma_{022}, \gamma_{032}) \equiv 0, & \mathbf{v}_3 &= (\gamma_{013}, \gamma_{023}, \gamma_{033}) \equiv 0, \\ \mathbf{p}_0 &= (\gamma_{230}, \gamma_{310}, \gamma_{120}) = 0, & \mathbf{p}_1 &= (\gamma_{231}, \gamma_{311}, \gamma_{121}) = 0, \\ \mathbf{p}_2 &= (\gamma_{232}, \gamma_{312}, \gamma_{122}) = (0, 0, \frac{\cos r}{\sin r}), & \mathbf{p}_3 &= (\gamma_{233}, \gamma_{313}, \gamma_{123}) = (0, \frac{\sin r}{\cos r}, 0), \end{aligned} \tag{5.3}$$

in the absence of an external source eq. (16.3) reads

$$\left( -i\partial_t + \alpha^1 \partial_r + \alpha^2 \frac{1}{\sin r} \partial_\phi + \alpha^3 \frac{1}{\cos r} \partial_z + \alpha^2 S_3 \frac{\cos r}{\sin r} + \alpha^3 S_2 \frac{\sin r}{\cos r} \right) \left| \begin{matrix} 0 \\ \mathbf{E} + ic\mathbf{B} \end{matrix} \right| = 0. \tag{5.4}$$

## 6 Separation of variables in $S_3$ , solutions at $m = 0$

Wave Maxwell operator from (5.4) commutes with the following ones:  $i\partial_t$ ,  $i\partial_\phi$ ,  $i\partial_z$ . Therefore, for a field function we get a substitution

$$\Psi = \begin{vmatrix} 0 \\ \mathbf{E} + ic\mathbf{B} \end{vmatrix} = e^{-i\omega t} e^{im\phi} e^{ikz} \begin{vmatrix} 0 \\ f_1(r) \\ f_2(r) \\ f_3(r) \end{vmatrix}. \quad (6.1)$$

Correspondingly, eq. (5.4) reads

$$\left( -\omega + \alpha^1 \frac{d}{dr} + \frac{im}{\sin r} \alpha^2 + \frac{ik}{\cos r} \alpha^3 + \frac{\cos r}{\sin r} \alpha^2 S_3 + \frac{\sin r}{\cos r} \alpha^3 S_2 \right) \begin{vmatrix} 0 \\ f_1(r) \\ f_2(r) \\ f_3(r) \end{vmatrix} = 0. \quad (6.2)$$

After simple calculation we get the following radial system:

$$\begin{aligned} \left( \frac{d}{dr} + \frac{\cos r}{\sin r} - \frac{\sin r}{\cos r} \right) f_1 + \frac{im}{\sin r} f_2 + \frac{ik}{\cos r} f_3 &= 0, \\ -\omega f_1 - \frac{ik}{\cos r} f_2 + \frac{im}{\sin r} f_3 &= 0, \\ -\omega f_2 - \left( \frac{d}{dr} - \frac{\sin r}{\cos r} \right) f_3 + \frac{ik}{\cos r} f_1 &= 0, \\ -\omega f_3 + \left( \frac{d}{dr} + \frac{\cos r}{\sin r} \right) f_2 - \frac{im}{\sin r} f_1 &= 0. \end{aligned} \quad (6.3)$$

Let us consider first the case  $m = 0$ , when the radial equations become more simple

$$\begin{aligned} \left( \frac{d}{dr} + \frac{\cos r}{\sin r} - \frac{\sin r}{\cos r} \right) f_1 + \frac{ik}{\cos r} f_3 &= 0, \\ f_1 &= -\frac{ik}{\omega \cos r} f_2, \\ -\omega f_2 - \left( \frac{d}{dr} - \frac{\sin r}{\cos r} \right) f_3 + \frac{ik}{\cos r} f_1 &= 0, \\ f_3 &= \frac{1}{\omega} \left( \frac{d}{dr} + \frac{\cos r}{\sin r} \right) f_2. \end{aligned} \quad (6.4)$$

Using 2nd and 4th equations, from the first one it follows

$$\begin{aligned} \left( \frac{d}{dr} + \frac{\cos r}{\sin r} - \frac{\sin r}{\cos r} \right) \frac{-ik}{\omega \cos r} f_2 + \frac{ik}{\cos r} \frac{1}{\omega} \left( \frac{d}{dr} + \frac{\cos r}{\sin r} \right) f_2 &= 0, \\ -\omega f_2 - \left( \frac{d}{dr} - \frac{\sin r}{\cos r} \right) \frac{1}{\omega} \left( \frac{d}{dr} + \frac{\cos r}{\sin r} \right) f_2 + \frac{ik}{\cos r} \frac{-ik}{\omega \cos r} f_2 &= 0, \end{aligned}$$

that is equivalent to the identity  $0 \equiv 0$  and the equation for  $f_2$ :

$$\frac{d^2}{dr^2} f_2 + \left( \frac{\cos r}{\sin r} - \frac{\sin r}{\cos r} \right) \frac{d}{dr} f_2 + \left( \omega^2 - 1 - \frac{1}{\sin^2 r} - \frac{k^2}{\cos^2 r} \right) f_2 = 0. \quad (6.5)$$



The latter can be simplified:

$$f_2(r) = \frac{1}{\sin r} E(r) , \quad \frac{d^2 E}{dr^2} - \frac{1}{\sin r \cos r} \frac{dE}{dr} + (\omega^2 - \frac{k^2}{\cos^2 r}) E = 0 ; \quad (6.6)$$

two concomitant functions are given by

$$f_1(r) = \frac{-ik}{\omega} \frac{1}{\cos r \sin r} E(r) , \quad f_3 = \frac{1}{\omega} \frac{1}{\sin r} \frac{d}{dr} E(r) .$$

Turning back to eq. (6.6), first let us consider a particular case when  $k^2 = \omega^2$ :

$$\frac{\sin r}{\cos r} \frac{d}{dr} \frac{\cos r}{\sin r} \frac{d}{dr} E + k^2 (1 - \frac{1}{\cos^2 r}) E = 0 ;$$

from whence it follows

$$\left( \frac{\cos r}{\sin r} \frac{d}{dr} \right) \left( \frac{\cos r}{\sin r} \frac{d}{dr} \right) E = k^2 E ,$$

so that

$$\frac{\cos r}{\sin r} \frac{d}{dr} = \frac{d}{dx} , \quad \implies \quad \frac{dr}{dx} = \frac{\cos r}{\sin r} ,$$

$$-dx = d \log \cos r , \quad x = \log(C \cos^{-1} r) , \quad C = \text{const}$$

and eq. (6.7) reads

$$\frac{d^2}{dx^2} E = k^2 E ,$$

which has two solutions

$$k^2 = \omega^2 , \quad E_{\pm} = e^{\mp kx} = E_0 (\cos r)^{\pm k} . \quad (6.7)$$

Among solutions

$$(\cos r)^{+k} , \quad \frac{1}{(\cos r)^k} , \quad r \in [0, \frac{\pi}{2}] \quad (6.8)$$

at  $k > 0$  the second one must be rejected because it turns to infinity as  $r \rightarrow \pi/2$ ; when for  $k < 0$  the first one must be rejected by analogous reason. Thus, physical solutions are

$$k^2 = \omega^2 , \quad k > 0 , \quad E = E_0 \cos^k r e^{-i(\omega t - kz)} ;$$

$$k^2 = \omega^2 , \quad k < 0 , \quad E = E_0 \cos^{-k} r e^{-i(\omega t - kz)} . \quad (6.9)$$

Because at  $r \neq 0, \pi/2$ , the values  $z = -\pi$  and  $z = +\pi$  determine one the same point in spherical space  $S_3$ , solutions (6.9) represent continuous function in  $S_3$  only if  $k$  takes on integer values:

$$k = \pm n , \quad n = 1, 2, 3, \dots ; \quad (6.10)$$

or in usual units

$$k = \pm \frac{\omega \rho}{c} = n, \quad \omega = \frac{c}{\rho} n, \quad n = 1, 2, 3, \dots \quad (6.11)$$

Turning again to eq. (6.6) let us construct one more special solution. To this end, considering approximate solution in the vicinity of the point  $r = 0$ :  $E = \sin^A r$ , we get  $A = 0, +2$ . In the same manner nearby the point  $r = \pi/2$  an approximate solution is  $E = \cos^B r$ ,  $B^2 = k^2$ . Let us demonstrate that there exist values  $k$  such that an exact solution can be constructed as follows (the case  $B = 0$  was considered above):

$$E = \sin^2 r \cos^B r. \quad (6.12)$$

With this substitution, eq. (6.6) gives

$$2 \cos^4 r - 2(B+1) \sin^2 r \cos^2 r - 3B \sin^2 r \cos^2 r + B(B-1) \sin^4 r - 2 \cos^2 r + B \sin^2 r - k^2 \sin^2 r + \omega^2 \sin^2 r \cos^2 r = 0.$$

With notation  $\cos^2 r = x$  it can be written as

$$2x^2 + (x - x^2)[-5B - 2 + \omega^2] + (B^2 - B)(1 - 2x + x^2) - 2x + B(1 - x) - k^2(1 - x) = 0,$$

or

$$x^2(4 + 4B - \omega^2 + B^2) + x(-4 - 4B + \omega^2 - B^2) + x^0(B^2 - k^2) = 0.$$

The latter is satisfied if

$$B^2 = k^2, \quad (B+2)^2 - \omega^2 = 0,$$

that is

$$\begin{aligned} B &= -2 + \omega, \quad -2 - \omega, \\ k &= \pm B, \quad E = \sin^2 r \cos^B r. \end{aligned} \quad (6.13)$$

Thus, the corresponding solutions of this type are

$$E = \sin^2 r \cos^B r e^{-i(\omega t - kz)}. \quad (6.14)$$

Solutions with negative  $B$  must be rejected because they give infinite electromagnetic field at the point  $r = \pi/2$ . Besides, periodicity requirement on  $z$  leads to  $k = \pm 1, \pm 2, \pm 3, \dots$

Therefore, the wave propagating in the positive direction is given by

$$\begin{aligned} B &= +k = +1, +2, +3, \dots, \\ k &= -2 + \omega, \quad \omega = 2 + k = 3, 4, 5, \dots, \\ E &= \sin^2 r \cos^k r e^{-i(\omega t - kz)}. \end{aligned} \quad (6.15)$$

In turn, the wave propagating in the negative direction is given by

$$\begin{aligned} B &= -k = +1, +2, +3, \dots, \\ -k &= -2 + \omega, \quad \omega = 2 - k = +3, +4, +5, \dots, \\ E &= \sin^2 r \cos^{-k} r e^{-i(\omega t - kz)}. \end{aligned} \quad (6.16)$$

Turning to the general equation (6.6), one may try to construct all other solutions of that type  $m = 0$  on the base of the following substitution:

$$E(r) = \sin^2 r \cos^B r F(r) ; \quad (6.17)$$

eq. (6.6) gives (let  $\cos^2 r = x$ )

$$4x(1-x) \frac{d^2}{dx^2} F + 4[1-3x+B(1-x)] \frac{d}{dx} F + \left[ -(2+5B) + \frac{2x}{1-x} + B(B-1) \frac{1-x}{x} - \frac{2}{1-x} + \frac{B}{x} - \frac{k^2}{x} + \omega^2 \right] F = 0 .$$

Requiring  $k^2 = B^2$ , for  $F$  we get the equation

$$4x(1-x) \frac{d^2}{dx^2} F + 4[(1+B) - (3+B)x] \frac{d}{dx} F - [(B+2)^2 - \omega^2] F = 0 ,$$

which is of hypergeometric type

$$z(1-z) F + [\gamma - (\alpha + \beta + 1)z] F' - \alpha\beta F = 0 , \\ k = \pm B , \quad \gamma = 1 + B , \quad \alpha = \frac{B+2-\omega}{2} , \quad \beta = \frac{B+2+\omega}{2} .$$

Thus, the general solution of the type  $m = 0$  takes the form

$$E = \sin^2 r \cos^B r F(\alpha, \beta, \gamma, \cos^2 r) e^{-i(\omega t - kz)} . \quad (6.18)$$

We are to separate single-valued and continuous functions in  $S_3$ .

For a wave propagating in the positive direction  $z$ :

$$k > 0 , \quad k = +1, +2, +3, \dots ; \quad (6.19)$$

the function  $E(r)$  is finite at  $r = \pi/2$  only if  $B = +k$ , besides polynomial solutions arise only if

$$\alpha = \frac{k+2-\omega}{2} = -n = 0, -1, -2, \dots \implies \\ \omega = k + 2(n+1) = N . \quad (6.20)$$

For a wave propagating in the positive direction  $z$ :

$$k < 0 , \quad -k = 1, 2, 3, \dots ; \quad (6.21)$$

the function  $E(r)$  is finite at  $r = \pi/2$  only if  $B = -k$ , additionally one must obtain polynomials which leads to

$$\alpha = \frac{-k+2-\omega}{2} = -n = 0, -1, -2, \dots \implies \\ \omega = -k + 2(n+1) = N . \quad (6.22)$$

All constructed solutions of the Maxwell equations are finite, single-valued, and continuous functions in spherical Riemann space  $S_3$ .

## 7 Maxwell solutions at $k = 0$

Turning to eqs. (6.3) at  $k = 0$ :

$$\begin{aligned}
\left(\frac{d}{dr} + \frac{\cos r}{\sin r} - \frac{\sin r}{\cos r}\right)f_1 + \frac{im}{\sin r} f_2 &= 0 , \\
f_1 &= \frac{im}{\omega \sin r} f_3 , \\
f_2 &= -\frac{1}{\omega} \left(\frac{d}{dr} - \frac{\sin r}{\cos r}\right) f_3 , \\
-\omega f_3 + \left(\frac{d}{dr} + \frac{\cos r}{\sin r}\right) f_2 - \frac{im}{\sin r} f_1 &= 0 .
\end{aligned} \tag{7.1}$$

With the use of second and third from first and fourth we get an identity  $0 \equiv 0$  and the following equation for  $f_3$ :

$$\frac{d^2}{dr^2} f_3 + \left(\frac{\cos r}{\sin r} - \frac{\sin r}{\cos r}\right) \frac{d}{dr} f_3 + \left(\omega^2 - 1 - \frac{1}{\cos^2 r} - \frac{k^2}{\sin^2 r}\right) f_3 = 0 . \tag{7.2}$$

which gives

$$f_3(r) = \frac{1}{\cos r} E(r) , \quad \frac{d^2 E}{dr^2} + \frac{1}{\sin r \cos r} \frac{dE}{dr} + \left(\omega^2 - \frac{m^2}{\sin^2 r}\right) E = 0 . \tag{7.3}$$

First, consider a particular case  $m^2 = \omega^2$ :

$$\frac{d^2 E}{dr^2} + \frac{1}{\sin r \cos r} \frac{dE}{dr} + m^2 \left(1 - \frac{1}{\sin^2 r}\right) E = 0 ; \tag{7.4}$$

with the solutions

$$(\sin r)^m , \quad \frac{1}{(\sin r)^m} , \quad r \in [0, \frac{\pi}{2}] . \tag{7.5}$$

Physical solutions are

$$\begin{aligned}
m^2 = \omega^2 , \quad m > 0 , \quad E &= E_0 \sin^m r e^{-i(\omega t - m\phi)} ; \\
m^2 = \omega^2 , \quad m < 0 , \quad E &= E_0 \sin^{-m} r e^{-i(\omega t - m\phi)} .
\end{aligned} \tag{7.6}$$

Performing analysis like in previous Section, we easily construct solutions:

$$\begin{aligned}
B &= +m = +1, +2, +3, \dots , \\
m &= -2 + \omega , \quad \omega = 2 + m = 3, 4, 5, \dots , \\
F_{02} &= \cos^2 r \sin^m r e^{-i(\omega t - m\phi)} .
\end{aligned} \tag{7.7}$$

and

$$\begin{aligned}
B &= -m = +1, +2, +3, \dots , \\
-m &= -2 + \omega , \quad \omega = 2 - m = +3, +4, +5, \dots , \\
F_{02} &= \cos^2 r \sin^{-m} r e^{-i(\omega t - mz)} .
\end{aligned} \tag{7.8}$$

All possible solutions of eq. (7.3) can be constructed on the base of a substitution:

$$E(r) = \cos^2 r \sin^B r F(r) . \quad (7.9)$$

and further (let  $\sin^2 r = x$ ) we get

$$m^2 = B^2 , \quad 4x(1-x) \frac{d^2}{dx^2} F + 4[(1+B) - (3+B)x] \frac{d}{dx} F - [(B+2)^2 - \omega^2] F = 0 ,$$

what is of hypergeometric type

$$\begin{aligned} z(1-z) F + [\gamma - (\alpha + \beta + 1)z] F' - \alpha\beta F &= 0 , \\ k = \pm B , \quad \gamma = 1 + B , \quad \alpha = \frac{B+2-\omega}{2} , \quad \beta = \frac{B+2+\omega}{2} . \end{aligned} \quad (7.10)$$

Thus, the Maxwell equations solutions of the type  $k = 0$  is given by

$$E = \cos^2 r \sin^B r F(\alpha, \beta, \gamma, \sin^2 r) e^{-i(\omega t - m\phi)} . \quad (7.11)$$

We are to separate physical waves.

$$\begin{aligned} m > 0 , \quad B = +m , \\ \alpha = \frac{m+2-\omega}{2} = -n \quad \implies \quad \omega = m + 2(n+1) = N . \end{aligned} \quad (7.12)$$

$$\begin{aligned} m < 0 , \quad B = -m , \\ \alpha = \frac{-m+2-\omega}{2} = -n \quad \implies \quad \omega = -m + 2(n+1) = N . \end{aligned} \quad (7.13)$$

## 8 Radial system at arbitrary $m, k$ , general solutions

Now let us solve radial equations in general case (6.3). The first equation reduces to the identity  $0 = 0$  when taking into account three remaining:

$$\begin{aligned} -\omega f_1 &= \frac{ik}{\cos r} f_2 - \frac{im}{\sin r} f_3 , \\ -\omega f_2 &= \left( \frac{d}{dr} - \frac{\sin r}{\cos r} \right) f_3 - \frac{ik}{\cos r} f_1 , \\ -\omega f_3 &= -\left( \frac{d}{dr} + \frac{\cos r}{\sin r} \right) f_2 + \frac{im}{\sin r} f_1 ; \end{aligned} \quad (8.1)$$

and the first equation takes the form

$$\begin{aligned} \left( \frac{d}{dr} + \frac{\cos r}{\sin r} - \frac{\sin r}{\cos r} \right) \left( \frac{ik}{\cos r} f_2 - \frac{im}{\sin r} f_3 \right) + \frac{im}{\sin r} \left[ \left( \frac{d}{dr} - \frac{\sin r}{\cos r} \right) f_3 - \frac{ik}{\cos r} f_1 \right] + \\ + \frac{ik}{\cos r} \left[ -\left( \frac{d}{dr} + \frac{\cos r}{\sin r} \right) f_2 + \frac{im}{\sin r} f_1 \right] = 0 . \end{aligned}$$

what is the identity  $0 = 0$ . The system (8.1) is simplified:

$$f_2 = \frac{1}{\sin r} F_2 , \quad f_3 = \frac{1}{\cos r} F_3 ,$$

so that

$$\begin{aligned}
-\omega f_1 &= i \frac{k F_2 - m F_3}{\sin r \cos r} , \\
-\omega \frac{F_2}{\sin r} &= \frac{1}{\cos r} \frac{dF_3}{dr} - \frac{ik}{\cos r} f_1 , \\
-\omega \frac{F_3}{\cos r} &= -\frac{1}{\sin r} \frac{dF_2}{dr} + \frac{im}{\sin r} f_1 .
\end{aligned} \tag{8.2}$$

Excluding  $f_1$ , we arrive at

$$\begin{aligned}
\left( \frac{\omega}{\cos r} \frac{d}{dr} + \frac{km}{\sin r \cos^2 r} \right) F_3 + \frac{1}{\sin r} \left( \omega^2 - \frac{k^2}{\cos^2 r} \right) F_2 &= 0 , \\
\left( \frac{\omega}{\sin r} \frac{d}{dr} - \frac{km}{\cos r \sin^2 r} \right) F_2 + \frac{1}{\cos r} \left( -\omega^2 + \frac{m^2}{\sin^2 r} \right) F_3 &= 0 .
\end{aligned} \tag{8.3}$$

With the use of a new variable  $y = (1 - \cos 2r)/2$  the system reads

$$\begin{aligned}
\left( 2\omega \frac{d}{dy} - \frac{km}{y(1-y)} \right) F_2 + \left( -\frac{\omega^2}{1-y} + \frac{m^2}{y(1-y)} \right) F_3 &= 0 , \\
\left( 2\omega \frac{d}{dy} + \frac{km}{y(1-y)} \right) F_3 + \left( +\frac{\omega^2}{y} - \frac{k^2}{y(1-y)} \right) F_2 &= 0 .
\end{aligned} \tag{8.4}$$

Instead of  $F_2, F_3$  let us introduce new functions by means of linear transformation with unit determinant  $\alpha N - \beta M = 1$ :

$$\begin{aligned}
F_2 &= \alpha(y) G_2 + \beta(y) G_3 , \\
F_3 &= M(y) G_2 + N(y) G_3 ,
\end{aligned} \tag{8.5}$$

and inverse given by

$$\begin{aligned}
G_2 &= N(y) F_2 - \beta(y) F_3 , \\
G_3 &= -M(y) F_2 + \alpha(y) F_3 .
\end{aligned} \tag{8.6}$$

Combining eqs. (8.4) we get

$$\begin{aligned}
&N \left( 2\omega \frac{d}{dy} - \frac{km}{y(1-y)} \right) F_2 + 2\omega \frac{dN}{dy} F_2 - 2\omega \frac{dN}{dy} F_2 + N \left( -\frac{\omega^2}{1-y} + \frac{m^2}{y(1-y)} \right) F_3 - \\
&-\beta \left( 2\omega \frac{d}{dy} + \frac{km}{y(1-y)} \right) F_3 - 2\omega \frac{d\beta}{dy} F_3 + 2\omega \frac{d\beta}{dy} F_3 - \beta \left( +\frac{\omega^2}{y} - \frac{k^2}{y(1-y)} \right) F_2 = 0 , \\
&-M \left( 2\omega \frac{d}{dy} - \frac{km}{y(1-y)} \right) F_2 - 2\omega \frac{dM}{dy} F_2 + 2\omega \frac{dM}{dy} F_2 - M \left( -\frac{\omega^2}{1-y} + \frac{m^2}{y(1-y)} \right) F_3 + \\
&+\alpha \left( 2\omega \frac{d}{dy} + \frac{km}{y(1-y)} \right) F_3 + 2\omega \frac{d\alpha}{dy} F_3 - 2\omega \frac{d\alpha}{dy} F_3 + \alpha \left( +\frac{\omega^2}{y} - \frac{k^2}{y(1-y)} \right) F_2 = 0 ,
\end{aligned} \tag{8.7}$$

from whence it follows that

$$\begin{aligned}
2\omega \frac{d}{dy} G_2 - N \frac{km}{y(1-y)} F_2 - 2\omega \frac{dN}{dy} F_2 + N \left( -\frac{\omega^2}{1-y} + \frac{m^2}{y(1-y)} \right) F_3 - \\
-\beta \frac{km}{y(1-y)} F_3 + 2\omega \frac{d\beta}{dy} F_3 - \beta \left( +\frac{\omega^2}{y} - \frac{k^2}{y(1-y)} \right) F_2 = 0 , \\
2\omega \frac{d}{dy} G_3 + M \frac{km}{y(1-y)} F_2 + 2\omega \frac{dM}{dy} F_2 - M \left( -\frac{\omega^2}{1-y} + \frac{m^2}{y(1-y)} \right) F_3 + \\
+\alpha \frac{km}{y(1-y)} F_3 - 2\omega \frac{d\alpha}{dy} F_3 + \alpha \left( +\frac{\omega^2}{y} - \frac{k^2}{y(1-y)} \right) F_2 = 0 .
\end{aligned} \tag{8.8}$$

Instead of  $F_2, F_3$  we substitute their expression through  $G_2, G_3$  according to (8.5):

$$\begin{aligned}
2\omega \frac{dG_2}{dy} + \left[ -(N\alpha + \beta M) \frac{km}{y(1-y)} - 2\omega \frac{dN}{dy} \alpha + NM \frac{-\omega^2 y + m^2}{y(1-y)} + \right. \\
\left. + 2\omega \frac{d\beta}{dy} M - \beta \alpha \frac{\omega^2(1-y) - k^2}{y(1-y)} \right] G_2 + \\
+ \left[ -2N\beta \frac{km}{y(1-y)} - 2\omega \frac{dN}{dy} \beta + N^2 \frac{-\omega^2 y + m^2}{y(1-y)} + \right. \\
\left. + 2\omega \frac{d\beta}{dy} N - \beta^2 \frac{\omega^2(1-y) - k^2}{y(1-y)} \right] G_3 = 0 ,
\end{aligned} \tag{8.9}$$

$$\begin{aligned}
2\omega \frac{dG_3}{dy} + \left[ (M\beta + \alpha N) \frac{km}{y(1-y)} + 2\omega \frac{dM}{dy} \beta - NM \frac{-\omega^2 y + m^2}{y(1-y)} - \right. \\
\left. - 2\omega \frac{d\alpha}{dy} N + \beta \alpha \frac{\omega^2(1-y) - k^2}{y(1-y)} \right] G_3 + \\
+ \left[ 2M\alpha \frac{km}{y(1-y)} + 2\omega \frac{dM}{dy} \alpha - M^2 \frac{-\omega^2 y + m^2}{y(1-y)} - \right. \\
\left. - 2\omega \frac{d\alpha}{dy} M + \alpha^2 \frac{\omega^2(1-y) - k^2}{y(1-y)} \right] G_2 = 0 ,
\end{aligned} \tag{8.10}$$

Let us assume that the transformation used is an orthogonal one:

$$\begin{aligned}
\alpha G_2 + \beta G_3 &= \cos A G_2 + \sin A G_3 , \\
M G_2 + N G_3 &= -\sin A G_2 + \cos A G_3 ,
\end{aligned} \tag{8.11}$$

then

$$\begin{aligned}
-2\omega \frac{dN}{dy} \alpha + 2\omega \frac{d\beta}{dy} M &= -2\omega [(\cos A)' \cos A + (\sin A)' \sin A] = 0, \\
-2\omega \frac{dN}{dy} \beta + 2\omega \frac{d\beta}{dy} N &= 2\omega [-(\cos A)' \sin A + (\sin A)' \cos A] = +2\omega A', \\
2\omega \frac{dM}{dy} \beta - 2\omega \frac{d\alpha}{dy} N &= 2\omega [-(\sin A)' \sin A - (\cos A)' \cos A] = 0, \\
2\omega \frac{dM}{dy} \alpha - 2\omega \frac{d\alpha}{dy} M &= 2\omega [-(\sin A)' \cos A + (\cos A)' \sin A] = -2\omega A'.
\end{aligned}$$

and

$$\begin{aligned}
N\alpha + \beta M &= \cos 2A, & 2N\beta &= \sin 2A, & 2M\alpha &= -\sin 2A, \\
\alpha\beta &= \sin A \cos A = \frac{1}{2} \sin 2A, & NM &= -\sin A \cos A = -\frac{1}{2} \sin 2A, \\
N^2 &= \cos^2 A, & \beta^2 &= \sin^2 A, & M^2 &= \sin^2 A, & \alpha^2 &= \cos^2 A,
\end{aligned}$$

Therefore, eqs. (8.9) and (8.10) take the form

$$\begin{aligned}
&2\omega \frac{dG_2}{dy} - \left[ \cos 2A \frac{km}{y(1-y)} + \frac{1}{2} \sin 2A \frac{-\omega^2 y + m^2 + \omega^2(1-y) - k^2}{y(1-y)} \right] G_2 + \\
&+ \left[ +2\omega A' - \sin 2A \frac{km}{y(1-y)} + \cos^2 A \frac{-\omega^2 y + m^2}{y(1-y)} - \sin^2 A \frac{\omega^2(1-y) - k^2}{y(1-y)} \right] G_3 = 0, \\
&2\omega \frac{dG_3}{dy} + \left[ \cos 2A \frac{km}{y(1-y)} + \frac{1}{2} \sin 2A \frac{-\omega^2 y + m^2 + \omega^2(1-y) - k^2}{y(1-y)} \right] G_3 + \\
&+ \left[ -2\omega A' - \sin 2A \frac{km}{y(1-y)} - \sin^2 A \frac{-\omega^2 y + m^2}{y(1-y)} + \cos^2 A \frac{\omega^2(1-y) - k^2}{y(1-y)} \right] G_2 = 0,
\end{aligned}$$

Supposing that the used linear transformation does not depend on coordinate  $y$ , we get more simple expressions:

$$\begin{aligned}
&2\omega \frac{dG_2}{dy} - \left[ \cos 2A \frac{km}{y(1-y)} + \frac{1}{2} \sin 2A \frac{-\omega^2 y + m^2 + \omega^2(1-y) - k^2}{y(1-y)} \right] G_2 + \\
&+ \frac{-2km \sin 2A + (1 + \cos 2A)[-\omega^2 y + m^2] - (1 - \cos 2A)[\omega^2(1-y) - k^2]}{2y(1-y)} G_3 = 0, \quad (8.12)
\end{aligned}$$

$$\begin{aligned}
&2\omega \frac{dG_3}{dy} + \left[ \cos 2A \frac{km}{y(1-y)} + \frac{1}{2} \sin 2A \frac{-\omega^2 y + m^2 + \omega^2(1-y) - k^2}{y(1-y)} \right] G_3 + \\
&+ \frac{-2km \sin 2A - (1 - \cos 2A)[-\omega^2 y + m^2] + (1 + \cos 2A)[\omega^2(1-y) - k^2]}{2y(1-y)} G_2 = 0, \quad (8.13)
\end{aligned}$$



Let

$$\cos 2A = 0, \quad 2A = \frac{\pi}{2} \quad \sin 2A = 1$$

then eqs. (8.12)–(8.13) read

$$\begin{aligned} \left( 2\omega \frac{d}{dy} - \frac{-\omega^2 y + \omega^2(1-y) + m^2 - k^2}{2y(1-y)} \right) G_2 + \frac{-\omega^2 + (m-k)^2}{2y(1-y)} G_3 &= 0, \\ \left( 2\omega \frac{d}{dy} + \frac{-\omega^2 y + \omega^2(1-y) + m^2 - k^2}{2y(1-y)} \right) G_3 + \frac{\omega^2 - (m+k)^2}{2y(1-y)} G_2 &= 0. \end{aligned} \quad (8.14)$$

It is remarkable that in the system produced the singularities are located at the points  $y = 0, 1, \infty$  only.

From (8.14) and excluding  $G_3$  one straightforwardly gets the equation for  $G_2$ :

$$\begin{aligned} G_3 &= -2\omega \frac{2y(1-y)}{-\omega^2 + (m-k)^2} \frac{dG_2}{dy} + \frac{-\omega^2 y + \omega^2(1-y) + m^2 - k^2}{-\omega^2 + (m-k)^2} G_2, \\ 4y(1-y) \frac{d^2 G_2}{dy^2} + 4(1-2y) \frac{dG_2}{dy} + \left( 2\omega + \omega^2 - \frac{m^2}{y(1-y)} + \frac{m^2 - k^2}{1-y} \right) G_2 &= 0 \end{aligned} \quad (8.15)$$

With the substitution  $G_2 = y^A(1-y)^B G(y)$ , eq. (8.15) takes the form

$$\begin{aligned} 4y(1-y) G'' + 4 [A(1-y) - By + A(1-y) - By + (1-2y)] G' + \\ + \left[ 4A(A-1) \frac{1}{y} + 4B(B-1) \frac{1}{1-y} - 4A(A-1) - 4B(B-1) - 8AB + \right. \\ \left. + 4(-2A - 2B + \frac{A}{y} + \frac{B}{1-y}) + 2\omega + \omega^2 - m^2(\frac{1}{y} + \frac{1}{1-y}) + \frac{m^2 - k^2}{1-y} \right] G &= 0 \end{aligned} \quad (8.16)$$

Requiring

$$\begin{aligned} 4A(A-1) + 4A - m^2 &= 0 \implies A = \pm \frac{1}{2} |m|, \\ 4B(B-1) + 4B - k^2 &= 0 \implies B = \pm \frac{1}{2} |k|; \end{aligned} \quad (8.17)$$

we arrive at

$$\begin{aligned} y(1-y) G'' + [2A + 1 - 2(A+B+1)y] G' - \\ - \left[ (A+B)(A+B+1) - \frac{\omega}{2}(\frac{\omega}{2} + 1) \right] G &= 0, \end{aligned}$$

what is of hypergeometric type

$$\begin{aligned} \gamma &= 2A + 1, \quad \alpha + \beta = 2A + 2B + 1, \\ \alpha\beta &= (A+B)(A+B+1) - \frac{\omega}{2}(\frac{\omega}{2} + 1), \end{aligned}$$

that is

$$\alpha = A + B - \frac{\omega}{2}, \quad \beta = A + B + 1 + \frac{\omega}{2}, \quad \gamma = 2A + 1. \quad (8.18)$$

The functions are finite on the sphere  $S_3$  only if

$$A = +\frac{1}{2} |m|, \quad B = +\frac{1}{2} |k|, \quad \alpha = -n = 0, -1, -2, \dots;$$

which leads to the frequency spectrum in the form :

$$\omega = 2(n + A + B) = 2n + |m| + |k|; \quad (8.19)$$

the parameters  $m$  and  $k$  are allowed to be integer only :  $m, k \in \{0, \pm 1, \pm 2, \dots\}$ . The function  $G_2(y)$  is

$$G_2(y) = M_2 y^{|m|/2} (1-y)^{|k|/2} F(-n, n+1+|m|+|k|, |m|+1; y). \quad (8.20)$$

In eqs. (8.14) we might exclude  $G_2$ :  $G_3$ :

$$G_2(y) = -2\omega \frac{2y(1-y)}{\omega^2 - (m+k)^2} \frac{dG_3}{dy} - \frac{-\omega^2 y + \omega^2(1-y) + m^2 - k^2}{\omega^2 - (m+k)^2} G_3, \\ 4y(1-y) \frac{d^2 G_3}{dy^2} + 4(1-2y) \frac{dG_3}{dy} + \left( -2\omega + \omega^2 - \frac{m^2}{y(1-y)} + \frac{m^2 - k^2}{1-y} \right) G_3 = 0 \quad (8.21)$$

With the use of substitution  $G_3 = y^A(1-y)^B F(y)$ , the equation for  $G_3$  reduces to

$$4y(1-y) F'' + 4[A(1-y) - By + A(1-y) - By + (1-2y)] F' + \\ + \left[ 4A(A-1) \frac{1}{y} + 4B(B-1) \frac{1}{1-y} - 4A(A-1) - 4B(B-1) - 8AB + \right. \\ \left. + 4(-2A - 2B + \frac{A}{y} + \frac{B}{1-y}) - 2\omega + \omega^2 - m^2(\frac{1}{y} + \frac{1}{1-y}) + \frac{m^2 - k^2}{1-y} \right] F = 0 \quad (8.22)$$

Requiring

$$4A(A-1) + 4A - m^2 = 0 \implies A = +\frac{1}{2} |m|, \\ 4B(B-1) + 4B - k^2 = 0 \implies B = +\frac{1}{2} |k|;$$

we arrive at a hypergeometric type equation

$$y(1-y) F'' + [2A+1 - 2(A+B+1)y] F' - \\ - \left[ (A+B)(A+B+1) - \frac{\omega}{2}(\frac{\omega}{2} - 1) \right] F = 0, \\ a = A + B + 1 - \frac{\omega}{2}, \quad b = A + B + \frac{\omega}{2}, \quad c = 2A + 1. \quad (8.23)$$

Further we get

$$\begin{aligned}
a &= A + B + 1 - \frac{\omega}{2} = -N, \quad N = 0, 1, 2, \dots, \\
\omega &= 2(A + B + 1 + N) = |m| + |k| + 2(1 + N), \quad \underline{N + 1 = n}, \\
G_3 &= M_3 y^{|m|/2} (1 - y)^{|k|/2} F(-n + 1, n + |m| + |k|, |m| + 1; y);
\end{aligned} \tag{8.24}$$

compare with (8.20).

It remains to find a relative factor in two functions  $G_2$  and  $G_3$ :

$$\begin{aligned}
G_2 &= M_2 y^{|m|/2} (1 - y)^{|k|/2} F(-n, n + 1 + |m| + |k|, |m| + 1; y), \\
G_3 &= M_3 y^{|m|/2} (1 - y)^{|k|/2} F(-n + 1, n + |m| + |k|, |m| + 1; y),
\end{aligned} \tag{8.25}$$

and the relationship (see (8.15))

$$G_3 [(m - k)^2 - \omega^2] = -4\omega y(1 - y) \frac{dG_2}{dy} + [m^2 - k^2 + \omega^2(1 - 2y)] G_2.$$

must hold. Using the expressions for  $G_2$   $G_3$  we get

$$\begin{aligned}
&(m - k - \omega) (m - k + \omega) \frac{M_3}{M_2} F_3(y) = \\
&= -4\omega \left[ \frac{|m|}{2} (1 - y) F_2(y) - \frac{|k|}{2} y F_2(y) + \right. \\
&\left. + y(1 - y) \frac{d}{dy} F_2(y) \right] + [m^2 - k^2 + \omega^2(1 - 2y)] F_2(y).
\end{aligned} \tag{8.26}$$

It is sufficient to consider this equation in the point  $y = 0$  only that results in

$$-(\omega + m - k) (\omega - m + k) \frac{M_3}{M_2} = (\omega - |m| - k) (\omega - |m| + k),$$

and therefore

$$\begin{aligned}
M_2 &= M (\omega + m - k)(\omega - m + k), \\
M_3 &= -M (\omega - |m| - k)(\omega - |m| + k).
\end{aligned} \tag{8.27}$$

Depending on the sign of  $m$  it may be rewritten in a simpler form:

$$\begin{aligned}
m > 0, \quad M_2 &= M(\omega - k + m), \quad M_3 = -M(\omega - k - m); \\
m < 0, \quad M_2 &= M(\omega + k - m), \quad M_3 = -M(\omega + k + m); \\
m = 0, \quad M_2 &= M, \quad M_3 = -M;
\end{aligned} \tag{8.28}$$

$M$  stands for a numerical constant.

## 9 Maxwell solutions in elliptical model

Let us consider the problem of Maxwell solutions in elliptical space  $S'_3$ . This space  $S'_3$  is a space of constant positive curvature also and differs from the spherical model in topological properties only:  $S_3$  is 1-connected,  $S'_3$  is a 2-connected. The question is on the role of these differences for electromagnetic field solutions.

To obtain explicit realizations for two models it is convenient to use relations known in the theory of unitary and orthogonal groups. To each point in  $S_3$  there exists corresponding element in unitary group  $SU(2)$ :

$$B = \sigma^0 n_0 - i \sigma^k n_k, \quad \det B = +1.$$

In turn, to each point in elliptic space  $S^3$  there exists corresponding element in  $SO(3)$  parameterized by Gibbs 3-vector [23]

$$0(\vec{c}) = I + 2 \frac{\vec{c}^\times + (\vec{c}^\times)^2}{(1 + \vec{c}^2)} \quad , \quad (\vec{c}^\times)_{kl} = -\epsilon_{klj} c_j \quad . \quad (9.1)$$

note that two infinite length vectors represent one the same point in  $S'_3$ :

$$\vec{c}_\infty^+ = +\infty \vec{c}_0, \quad \vec{c}_\infty^- = -\infty \vec{c}_0, \quad \vec{c}_0^2 = 1, \quad 0(\vec{c}^{\pm \infty}) = I + 2(\vec{c}_0^\times)^2.$$

Mapping  $2 \rightarrow 1$  from  $SU(2)$  to  $SO(3)$  is given by

$$\{+n_a; -n_a\} \rightarrow \vec{c} = \frac{\vec{n}}{n_0}.$$

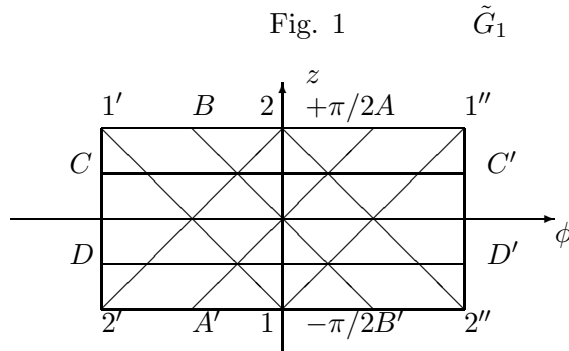
Cylindric coordinates  $(\rho, \phi, z)$  in elliptic space can be defined by the relations

$$\begin{aligned} c_1 &= \frac{\tan \rho}{\cos \rho} \cos z, \quad c_2 = \frac{\tan \rho}{\cos z} \sin z, \quad c_3 = \tan z, \\ \tilde{G}, \quad \rho &\in [0, \pi/2], \quad \phi \in [-\pi, +\pi], \quad z \in [-\pi/2, +\pi/2]. \end{aligned} \quad (9.2)$$

Additionally, we must define such an identification rule on the boundary of the region  $\tilde{G}$ , which agrees with the identification rule for vectors  $\vec{c}_\infty^+$  and  $\vec{c}_\infty^-$ . To this end, it is convenient to divide the region  $\tilde{G}$  into three parts:

$$\tilde{G}_1 = \tilde{G}(\rho \neq 0, \pi/2) \ , \quad \tilde{G}_2 = \tilde{G}(\rho = 0) \ , \quad \tilde{G}_3 = \tilde{G}(\rho = \pi/2) \ .$$

For the region  $\tilde{G}_1$  identification is given by (for more detail see [...])



each pair  $(A, A')$ ,  $(B, B')$  and so on represents one the same point in elliptical model  $S'_3$ ; also  $(1, 1', 1'')$  and  $(2, 2', 2'')$  correspond to one respective point in  $S'_3$ .

Now we should find what of above constructed Maxwell solutions in case of spherical model  $S_3$  will be single-valued ones when considering elliptical model  $S'_3$  (here we examine only points parameterized by the region  $\tilde{G}_1$ ). Evidently, it is sufficient to examine the behavior of the vector  $f = e^{im\phi} e^{ikz}$ . From the relations

$$f(C) = f(C'), \quad f(D) = f(D'), \quad \dots$$

no additional restrictions arise besides that  $M$  and  $K$  to be integer. Equations  $f(1) = f(1') = f(1'')$  give

$$e^{-ik(\pi/2)} = e^{-im\pi} e^{+ik(\pi/2)} = e^{+im\pi} e^{+ik(\pi/2)},$$

whence it follows that

$$e^{i2m\pi} = 1, \quad e^{i(k-m)\pi} = 1, \quad e^{i(k+m)\pi} = 1;$$

and therefore,  $(k - m)$  and  $(k + m)$  must be even. The same results follows from consideration of points  $(2, 2', 2'')$ .

Therefore, Maxwell solutions constructed above in spherical space will be single-valued solutions in elliptical space (in region  $\tilde{G}_1$ ) only if  $m$  and  $k$  are both even, or both odd; correspondingly,  $\omega$  parameter  $N$  in the expression for the frequency spectrum (see. (8.19)) given by

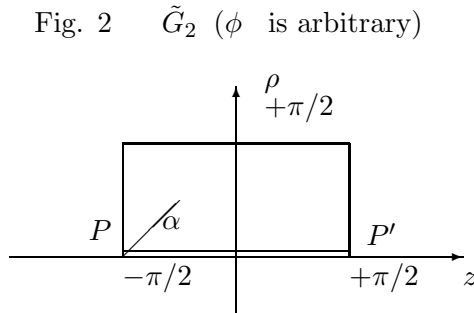
$$\omega = 2n + |m| + |k| = N \tag{9.3}$$

takes on even values:  $N = 0, 2, 4, 6, \dots$

Now let us consider the behavior of the above mentioned Maxwell solutions in the remaining regions  $\tilde{G}_2$  and  $\tilde{G}_3$ .

First, let us specify the case of  $\tilde{G}_2$  and consider the vicinity of the point  $P$  (see Fig. 2):

$$\rho = (\pi/2 \tan \alpha + z \tan \alpha) \implies \{z = -\pi/2 + \delta, \rho = \delta \tan \alpha\};$$



and

$$c_1 = \frac{\tan(\delta \tan \alpha)}{\sin(\delta \tan \alpha)} \frac{\cos \phi}{\sin \delta}, \quad c_2 = \frac{\tan(\delta \tan \alpha)}{\sin(\delta \tan \alpha)} \frac{\sin \phi}{\sin \delta}, \quad c_3 = -\frac{\cos \delta}{\sin \delta};$$

so in the limit  $\delta \rightarrow 0$  ( $\alpha \neq 0$ ) we get

$$\tilde{G}_2, \quad P, \quad \vec{c} = +\infty(0, 0, -1). \quad (9.4)$$

This means that coordinate  $\phi$  is "mute" at the point  $(\rho = 0, z = -\pi/2, \phi)$ .

For another point  $P' = (\rho = 0, z = +\pi/2, \phi)$  we get similar result:

$$\tilde{G}_2, \quad P', \quad \vec{c} = -\infty(0, 0, -1). \quad (9.5)$$

Thus,  $P$  and  $P'$  represent one the same point in elliptical space. Compare  $\Phi_{\omega mk}$  at these two points  $P$  and  $P'$ :

$$\Phi_{\omega mk} \sim e^{im\phi} e^{ikz} (\sin r)^{|m|} (\cos r)^{|k|} F(A, B, C; \sin^2 r);$$

and

$$\begin{aligned} \Phi_{\omega mk}(P) &\sim \begin{cases} 0, & \text{if } m \neq 0; \\ e^{+ik\pi/2} F(A, B, C; 0), & \text{if } m = 0; \end{cases} \\ \Phi_{\omega mk}(P') &\sim \begin{cases} 0, & \text{if } m \neq 0; \\ e^{-ik\pi/2} F(A, B, C; 0), & \text{if } m = 0; \end{cases} \end{aligned} \quad (9.6)$$

because  $k$  is even (at  $m = 0$ ) we have the identity  $\Phi_{\omega mk}(P) = \Phi_{\omega mk}(P')$ . In remaining part of the region  $\tilde{G}_2$ , points of elliptical model are parameterized according to

$$\begin{aligned} (0; \phi; z \neq 0, \pi \neq 2) &\implies \\ \vec{c} = (0, 0, \tan z), \quad \tilde{G}_2 \quad (\phi - \text{"mute" variable}). \end{aligned} \quad (9.7)$$

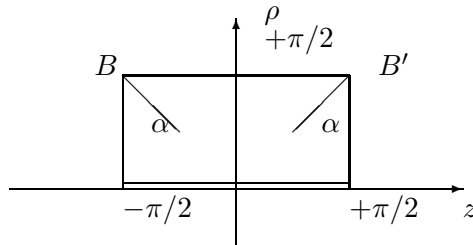
and the function  $\Phi_{\omega mk}$

$$G_2, \quad \phi_{\omega mk} \sim \begin{cases} 0, & \text{if } m \neq 0; \\ e^{ikz} F(A, B, C; 0), & \text{if } m = 0, \end{cases} \quad (9.8)$$

is single-valued and continuous in that part of  $\tilde{G}_2$  of the elliptic space.

Finally, let us consider the region  $\tilde{G}_3$ :

Fig. 3  $\tilde{G}_3$  ( $\phi$  is arbitrary)



In the vicinity of  $B$  we have

$$\rho = \left[ -z \tan \alpha + \left( \frac{\pi}{2} - \frac{\pi}{2} \tan \alpha \right) \right] \rightarrow \{ z = \left( -\frac{\pi}{2} + \delta \right), \rho = \left( \frac{\pi}{2} - \delta \tan \alpha \right) \};$$

that results in

$$c_1 = \frac{\cos(\delta \tan \alpha)}{\sin(\delta \tan \alpha)} \frac{\cos \phi}{\sin \delta}, \quad c_2 = \frac{\cos(\delta \tan \alpha)}{\sin(\delta \tan \alpha)} \frac{\sin \phi}{\sin \delta}, \quad c_3 = -\frac{\cos \delta}{\sin \delta};$$

from whence in the limit  $\delta \rightarrow 0$  (as  $\alpha \neq \pi/2$ ) it follows

$$B : \quad \vec{c} = \infty \frac{1}{\tan \alpha} (\cos \phi, \sin \phi, 0) = \infty (\cos \phi, \sin \phi, 0).$$

Analogously, in the vicinity of  $B'$  we get

$$B' : \quad \vec{c} = \infty (\cos \phi, \sin \phi, 0).$$

Comparing function at points  $B$  and  $B'$ :

$$\begin{aligned} \Phi_{\omega mk}(B) &\sim \begin{cases} 0, & \text{if } k \neq 0; \\ e^{+im\pi/2} F(A, B, C; 1), & \text{if } k = 0; \end{cases} \\ \Phi_{\omega mk}(B') &\sim \begin{cases} 0, & \text{if } k \neq 0; \\ e^{-im\pi/2} F(A, B, C; 1), & \text{if } k = 0; \end{cases} \end{aligned} \quad (9.9)$$

because  $m$  is even, the equality  $\Phi_{\omega mk}(B) = \Phi_{\omega mk}(B')$  holds. In remaining part of  $\tilde{G}_3$ :

$$\begin{aligned} \tilde{G}_3 : \quad &(\pi/2, \phi, z \neq -\pi/2, +\pi/2) \implies \\ &\vec{c} = \frac{\infty}{\cos z} (\cos \phi, \sin \phi, 0) = \infty (\cos \phi, \sin \phi, 0). \end{aligned} \quad (9.10)$$

$z$  is "mute" coordinate, therefore the solutions

$$\tilde{G}_3 = \begin{cases} 0, & \text{if } k \neq 0, \\ e^{imz} F(A, B, C; 1), & \text{if } k = 0, \end{cases} \quad (9.11)$$

represent single-valued and continuous functions in that part of elliptic space.

Thus, all functions  $\Phi_{\omega mk}(\rho, \phi, z)$  at  $\omega = N = 0, 2, 4, \dots$  are single-valued and continuous in elliptic space, and they represent physical solutions for Maxwell equation in this space, whereas all remaining functions  $\Phi_{\omega mk}(\rho, \phi, z)$ ,  $N = 1, 3, \dots$  should be rejected as non-single-valued and discontinuous in  $S'_3$  space.

## 10 Cylindric coordinate and tetrad in Lobachevsky space $H_3$ ,

Let us consider Maxwell equations in cylindric coordinate [22] of hyperbolic Lobachevsky space  $H_3$ :

$$\begin{aligned} n_1 &= \sinh r \cos \phi, \quad n_2 = \sinh r \sin \phi, \quad n_3 = \cosh r \sinh z, \quad n_4 = \cosh r \cosh z; \\ dS^2 &= dt^2 - dr^2 - \sinh^2 r d\phi^2 - \cosh^2 r dz^2, \quad x^\alpha = (t, r, \phi, z), \\ e_{(a)}^\beta(y) &= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sinh^{-1} r & 0 \\ 0 & 0 & 0 & \cosh^{-1} r \end{vmatrix}, \quad e_{(a)\beta}(y) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -\sinh r & 0 \\ 0 & 0 & 0 & -\cosh r \end{vmatrix}; \end{aligned} \quad (10.1)$$

where  $(r, \phi, z)$  run within

$$r \in [0, +\infty) , \quad \phi \in [0, 2\pi] , \quad z \in (-\infty, +\infty) .$$

Christoffel symbols are

$$\begin{aligned} \Gamma_{\beta\sigma}^0 = 0 , \quad \Gamma_{00}^i = 0 , \quad \Gamma_{0j}^i = 0 , \quad \Gamma_{jk}^z = \begin{vmatrix} 0 & 0 & \frac{\sinh r}{\cosh r} \\ 0 & 0 & 0 \\ \frac{\sinh r}{\cosh r} & 0 & 0 \end{vmatrix} , \\ \Gamma_{jk}^r = \begin{vmatrix} 0 & 0 & 0 \\ 0 & -\sinh r \cosh r & 0 \\ 0 & 0 & -\sinh r \cosh r \end{vmatrix} , \Gamma_{jk}^\phi = \begin{vmatrix} 0 & \frac{\cosh r}{\sinh r} & 0 \\ \frac{\cosh r}{\sinh r} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} . \end{aligned}$$

Derivatives of tetrad vectors read

$$\begin{aligned} e_{(0)\beta;\alpha} &= \frac{\partial e_{(0)\beta}}{\partial x^\alpha} - \Gamma_{\alpha\beta}^\sigma e_{(0)\sigma} = 0 , \\ e_{(1)\beta;\alpha} &= \frac{\partial e_{(1)\beta}}{\partial x^\alpha} - \Gamma_{\alpha\beta}^r e_{(1)r} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\sinh r \cosh r & 0 \\ 0 & 0 & 0 & -\sinh r \cosh r \end{vmatrix} , \\ e_{(2)\beta;\alpha} &= \frac{\partial e_{(2)\beta}}{\partial x^\alpha} - \Gamma_{\alpha\beta}^\phi e_{(2)\phi} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \cosh r & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} , \\ e_{(3)\beta;\alpha} &= \frac{\partial e_{(3)\beta}}{\partial x^\alpha} - \Gamma_{\alpha\beta}^z e_{(3)z} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sinh r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} . \end{aligned} \tag{10.2}$$

Ricci rotation coefficients are

$$\gamma_{011} = \gamma_{021} = \gamma_{031} = 0 , \quad \gamma_{012} = \gamma_{022} = \gamma_{032} = 0 , \quad \gamma_{013} = \gamma_{023} = \gamma_{033} = 0 ,$$

and

$$\begin{aligned} \gamma_{231} &= 0 , \quad \gamma_{311} = 0 , \quad \gamma_{121} = 0 , \\ \gamma_{232} &= 0 , \quad \gamma_{312} = 0 , \quad \gamma_{122} = \frac{\cosh r}{\sinh r} , \\ \gamma_{233} &= 0 , \quad \gamma_{313} = -\frac{\sinh r}{\cosh r} , \quad \gamma_{123} = 0 . \end{aligned}$$



With relations

$$\begin{aligned}
e_{(0)}^\rho \partial_\rho &= \partial_{(0)} = \partial_t, & e_{(1)}^\rho \partial_\rho &= \partial_{(1)} = \partial_r, \\
e_{(2)}^\rho \partial_\rho &= \partial_{(2)} = \frac{1}{\sinh r} \partial_\phi, & e_{(3)}^\rho \partial_\rho &= \partial_{(3)} = \frac{1}{\cosh r} \partial_z, \\
\mathbf{v}_0 &= (\gamma_{010}, \gamma_{020}, \gamma_{030}) \equiv 0, & \mathbf{v}_1 &= (\gamma_{011}, \gamma_{021}, \gamma_{031}) \equiv 0, \\
\mathbf{v}_2 &= (\gamma_{010}, \gamma_{022}, \gamma_{032}) \equiv 0, & \mathbf{v}_3 &= (\gamma_{013}, \gamma_{023}, \gamma_{033}) \equiv 0, \\
\mathbf{p}_0 &= (\gamma_{230}, \gamma_{310}, \gamma_{120}) = 0, & \mathbf{p}_1 &= (\gamma_{231}, \gamma_{311}, \gamma_{121}) = 0, \\
\mathbf{p}_2 &= (\gamma_{232}, \gamma_{312}, \gamma_{122}) = (0, 0, \frac{\cosh r}{\sinh r}), \\
\mathbf{p}_3 &= (\gamma_{233}, \gamma_{313}, \gamma_{123}) = (0, -\frac{\sinh r}{\cosh r}, 0),
\end{aligned}$$

Maxwell equation (16.3) reads

$$\left( -i\partial_t + \alpha^1 \partial_r + \alpha^2 \frac{1}{\sinh r} \partial_\phi + \alpha^3 \frac{1}{\cosh r} \partial_z + \alpha^2 S_3 \frac{\cosh r}{\sinh r} - \alpha^3 S_2 \frac{\sinh r}{\cosh r} \right) \begin{vmatrix} 0 \\ \mathbf{E} + ic\mathbf{B} \end{vmatrix} = 0 \quad (10.3)$$

## 11 Separation of variables in space $H_3$

Maxwell matrix operator in (10.3) commutes with three operators:  $i\partial_t$ ,  $i\partial_\phi$ ,  $i\partial_z$ ; therefore solutions can be constructed on the base of the following substitution

$$\Psi = \begin{vmatrix} 0 \\ \mathbf{E} + ic\mathbf{B} \end{vmatrix} = e^{-i\omega t} e^{im\phi} e^{ikz} \begin{vmatrix} 0 \\ f_1(r) \\ f_2(r) \\ f_3(r) \end{vmatrix}. \quad (11.1)$$

and eq. (10.3) takes the form

$$\left( -\omega + \alpha^1 \frac{d}{dr} + \frac{im}{\sinh r} \alpha^2 + \frac{ik}{\cosh r} \alpha^3 + \frac{\cosh r}{\sinh r} \alpha^2 S_3 - \frac{\sinh r}{\cosh r} \alpha^3 S_2 \right) \begin{vmatrix} 0 \\ f_1(r) \\ f_2(r) \\ f_3(r) \end{vmatrix} = 0. \quad (11.2)$$

After simple calculations we get the radial system in the form

$$\begin{aligned}
\left( \frac{d}{dr} + \frac{\cosh r}{\sinh r} + \frac{\sinh r}{\cosh r} \right) f_1 + \frac{im}{\sinh r} f_2 + \frac{ik}{\cosh r} f_3 &= 0, \\
-\omega f_1 - \frac{ik}{\cosh r} f_2 + \frac{im}{\sinh r} f_3 &= 0, \\
-\omega f_2 - \left( \frac{d}{dr} + \frac{\sinh r}{\cosh r} \right) f_3 + \frac{ik}{\cosh r} f_1 &= 0, \\
-\omega f_3 + \left( \frac{d}{dr} + \frac{\cosh r}{\sinh r} \right) f_2 - \frac{im}{\sinh r} f_1 &= 0.
\end{aligned} \quad (11.3)$$

## 12 Solutions at $m = 0$ , in space $H_3$

First, let  $m = 0$  – then

$$\begin{aligned} \left(\frac{d}{dr} + \frac{\cosh r}{\sinh r} + \frac{\sinh r}{\cosh r}\right)f_1 + \frac{ik}{\cosh r} f_3 &= 0, \\ f_1 &= \frac{-ik}{\omega \cosh r} f_2, \\ -\omega f_2 - \left(\frac{d}{dr} + \frac{\sinh r}{\cosh r}\right) f_3 + \frac{ik}{\cosh r} f_1 &= 0, \\ f_3 &= \frac{1}{\omega} \left(\frac{d}{dr} + \frac{\cosh r}{\sinh r}\right) f_2. \end{aligned} \quad (12.1)$$

With the help of 2nd and 4th equation from the 1st and 3rd it follows the identity  $0 = 0$  and equation for  $f_2$ :

$$f_2(r) = \frac{1}{\sinh r} E(r), \quad \frac{d^2 E}{dr^2} - \frac{1}{\sinh r \cosh r} \frac{dE}{dr} + \left(\omega^2 - \frac{k^2}{\cosh^2 r}\right) E = 0; \quad (12.2)$$

besides

$$f_1(r) = \frac{-ik}{\omega} \frac{1}{\cosh r \sinh r} E(r), \quad f_3 = \frac{1}{\omega} \frac{1}{\sinh r} \frac{d}{dr} E(r).$$

Eq. (12.2) can be resolved very easily when  $k^2 = \omega^2$ :

$$\frac{\sinh r}{\cosh r} \frac{d}{dr} \frac{\cosh r}{\sinh r} \frac{d}{dr} E + k^2 \left(1 - \frac{1}{\cosh^2 r}\right) E = 0;$$

from whence it follows

$$\left(\frac{\cosh r}{\sinh r} \frac{d}{dr}\right) \left(\frac{\cosh r}{\sinh r} \frac{d}{dr}\right) E = -k^2 E. \quad (12.3)$$

With the help of a new variable

$$\begin{aligned} \frac{\cosh r}{\sinh r} \frac{d}{dr} &= \frac{d}{dx}, \quad \Rightarrow \quad \frac{dr}{dx} = \frac{\cosh r}{\sinh r}, \\ dx &= d \log \cosh r, \quad x = \log(C \cosh r), \quad C = \text{const}. \end{aligned}$$

we arrive at

$$\frac{d^2}{dx^2} E = -k^2 E, \quad E = e^{ikx} = \text{const} (\cosh r)^{ik}, \quad k = \pm \omega;$$

Two constructed solutions are conjugated:

$$\begin{aligned} (\cosh r)^{+ik} &= (e^{\log \cosh r})^{+ik} = \cos(k \log \cosh r) + i \sin(k \log \cosh r), \\ (\cosh r)^{-ik} &= [e^{\log \cosh r}]^{-ik} = \cos(k \log \cosh r) - i \sin(k \log \cosh r), \end{aligned}$$

therefore one can separate two independent real ones:

$$E_+(r) = \cos[k_0 \log \cosh r], \quad E_-(r) = \sin[k_0 \log \cosh r]. \quad (12.4)$$

In the limit of vanishing curvature they reduces to the known ones :

$$\begin{aligned} r \rightarrow 0, \quad E_+(r) &= \cos(k_0 \log \cosh r) \longrightarrow +1; \\ r \rightarrow 0, \quad E_-(r) &= \sin(k_0 \log \cosh r) \longrightarrow \text{const } r^2; \end{aligned} \quad (12.5)$$

these satisfies to the equation in the flat space-time

$$\frac{d}{dr} \frac{1}{r} \frac{d}{dr} E(r) = 0, \quad \implies \quad E(r) \sim 1, r^2.$$

In contrast to the flat space, in Lobachevsky model the waves (12.4) are both oscillating at infinity.

Let us show that (12.2) has another simple exact solution in the form

$$E = \sinh^2 r \cosh^B r. \quad (12.6)$$

Indeed, substitution of (12.6) into (12.2) leads to

$$\begin{aligned} 2 \cosh^4 r + 2(B+1) \sinh^2 r \cosh^2 r + 3B \sinh^2 r \cosh^2 r + B(B-1) \sinh^4 r - \\ - 2 \cosh^2 r - B \sinh^2 r - k^2 \sinh^2 r + \omega^2 \sinh^2 r \cosh^2 r = 0. \end{aligned}$$

With notation  $\cosh^2 r = x$  it reads

$$x^2 (4 + 4B + \omega^2 + B^2) + x (-4 - 4B - \omega^2 - B^2) + x^0 (B^2 + k^2) = 0.$$

The latter is satisfied if

$$B^2 = -k^2, \quad (B+2)^2 + \omega^2 = 0, \quad (12.7)$$

that is

$$B = -2 + i\omega, \quad -2 - i\omega, \quad k = \pm iB = \begin{cases} \mp (2i + \omega), \\ \mp (2i - \omega), \end{cases} \quad (12.8)$$

Corresponding solutions look as

$$E(t, r, z) = E_0 \sinh^2 r \cosh^B r e^{-i(\omega t - kz)}. \quad (12.9)$$

their real and imaginary parts are given by

$$\begin{aligned} B = -2 + i\omega, \quad k = iB = -2i - \omega, \\ E(t, r, z) = E_0 \sinh^2 r \cosh^{-2+i\omega} r e^{i(-2zi - \omega z - \omega t)} = E_0 \tanh^2 r \cosh^{i\omega} r e^{2z} e^{i(-\omega z - \omega t)} = \\ = E_0 \tanh^2 r e^{2z} [\cos(\omega \log \cosh r) + i \sin(\omega \log \cosh r)] [\cos(-\omega z - \omega t) + i \sin(-\omega z - \omega t)] = \\ = E_0 \tanh^2 r e^{2z} \cos(\omega \log \cosh r - \omega z - \omega t) + i E_0 \tanh^2 r e^{2z} \sin(\omega \log \cosh r - \omega z - \omega t) \\ B = -2 + i\omega, \quad k = -iB = 2i + \omega, \\ E(t, r, z) = E_0 \sinh^2 r \cosh^{-2+i\omega} r e^{i(2zi + \omega z - \omega t)} = E_0 \tanh^2 r \cosh^{i\omega} r e^{-2z} e^{i(\omega z - \omega t)} = \\ = E_0 \tanh^2 r e^{-2z} [\cos(\omega \log \cosh r) + i \sin(\omega \log \cosh r)] [\cos(\omega z - \omega t) + i \sin(\omega z - \omega t)] = \\ = E_0 \tanh^2 r e^{-2z} \cos(\omega \log \cosh r + \omega z - \omega t) + i E_0 \tanh^2 r e^{-2z} \sin(\omega \log \cosh r + \omega z - \omega t) \end{aligned}$$

$$B = -2 - i\omega, \quad k = iB = -2i + \omega,$$

$$\begin{aligned} E(t, r, z) &= E_0 \sinh^2 r \cosh^{-2-i\omega} r e^{i(-2zi+\omega z-\omega t)} = E_0 \tanh^2 r \cosh^{-i\omega} r e^{2z} e^{i(\omega z-\omega t)} = \\ &= E_0 \tanh^2 r e^{2z} [\cos(-\omega \log \cosh r) + i \sin(-\omega \log \cosh r)] [\cos(\omega z - \omega t) + i \sin(\omega z - \omega t)] = \\ &= E_0 \tanh^2 r e^{2z} \cos(-\omega \log \cosh r + \omega z - \omega t) + i E_0 \tanh^2 r e^{2z} \sin(-\omega \log \cosh r + \omega z - \omega t) \end{aligned}$$

$$B = -2 - i\omega, \quad k = -iB = 2i - \omega,$$

$$\begin{aligned} E(t, r, z) &= E_0 \sinh^2 r \cosh^{-2-i\omega} r e^{i(2zi-\omega z-\omega t)} = E_0 \tanh^2 r \cosh^{-i\omega} r e^{-2z} e^{i(-\omega z-\omega t)} = \\ &= E_0 \tanh^2 r e^{-2z} [\cos(-\omega \log \cosh r) + i \sin(-\omega \log \cosh r)] [\cos(-\omega z - \omega t) + i \sin(-\omega z - \omega t)] = \\ &= E_0 \tanh^2 r e^{-2z} \cos(-\omega \log \cosh r - \omega z - \omega t) + i E_0 \tanh^2 r e^{-2z} \sin(-\omega \log \cosh r - \omega z - \omega t). \end{aligned}$$

Physical sense of these waves is not clear. More insight can be reached when constructing all possible solutions of eq. (12.2) through the substitution

$$E = \sinh^2 r \cosh^B r F(r).$$

In that way, from (12.2) we arrive at an equation of hypergeometric type (if  $x = \sinh r$ ,  $k^2 + B^2 = 0$ )

$$4x(1-x) \frac{d^2}{dx^2} F + 4[(1+B) - (3+B)x] \frac{d}{dx} F - [(B+2)^2 + \omega^2] F = 0, \quad (12.10)$$

with

$$\gamma = 1 + B, \quad \alpha + \beta = 2 + B, \quad \alpha\beta = \frac{(B+2)^2 + \omega^2}{4},$$

that is

$$B = \pm i k, \quad \gamma = 1 + B, \quad \alpha = \frac{B+2-i\omega}{2}, \quad \beta = \frac{B+2+i\omega}{2}, \quad (12.11)$$

and

$$E(t, r, z) = \sinh^2 r \cosh^B r F(\alpha, \beta, \gamma, \cosh^2 r) e^{-i(\omega t - kz)}. \quad (12.12)$$

Evidently, above constructed solutions (12.7) – (12.9) can be obtained from the general relations (12.11) – (12.12), if one demands  $\alpha = 0$  or  $\beta = 0$ . However, no physical ground exists to impose such (polynomial) restrictions in the case of Lobachevsky space. Instead, the complete electromagnetic basis should include waves spreading along  $z$ , with real parameters  $k$ .

### 13 Solutions at $k = 0$ in space $H_3$

Now let us turn to eqs. (11.3) with  $k = 0$ :

$$\begin{aligned} \left( \frac{d}{dr} + \frac{\cosh r}{\sinh r} + \frac{\sinh r}{\cosh r} \right) f_1 + \frac{im}{\sinh r} f_2 &= 0, \\ f_1 &= \frac{im}{\omega \sinh r} f_3, \\ f_2 &= -\frac{1}{\omega} \left( \frac{d}{dr} + \frac{\sinh r}{\cosh r} \right) f_3, \\ -\omega f_3 + \left( \frac{d}{dr} + \frac{\cosh r}{\sinh r} \right) f_2 - \frac{im}{\sinh r} f_1 &= 0. \end{aligned} \quad (13.1)$$

The first and fourth equations gives

$$\begin{aligned} \left(\frac{d}{dr} + \frac{\cosh r}{\sinh r} + \frac{\sinh r}{\cosh r}\right) \frac{im}{\omega \sinh r} f_3 + \frac{im}{\sinh r} \left(-\frac{1}{\omega} \left(\frac{d}{dr} + \frac{\sinh r}{\cosh r}\right) f_3\right) &= 0, \\ -\omega f_3 + \left(\frac{d}{dr} + \frac{\cosh r}{\sinh r}\right) \left(-\frac{1}{\omega} \left(\frac{d}{dr} + \frac{\sinh r}{\cosh r}\right) f_3\right) - \frac{im}{\sinh r} \frac{im}{\omega \sinh r} f_3 &= 0. \end{aligned} \quad (13.2)$$

they reduces respectively to the identity  $0 \equiv 0$  and

$$f_3(r) = \frac{1}{\cosh r} E(r); \quad \frac{d^2 E}{dr^2} + \frac{1}{\sinh r \cosh r} \frac{dE}{dr} + \left(\omega^2 - \frac{m^2}{\sinh^2 r}\right) E = 0. \quad (13.3)$$

In variable  $y = -\sinh^2 r$  it is rewritten as

$$-4y(1-y) \frac{d^2}{dy^2} E - 4(1-y) \frac{d}{dy} E + \left(\omega^2 + \frac{m^2}{y}\right) E = 0,$$

that is solved in hypergeometric functions  $E = y^a(1-y)^b Y(y)$ :

$$\begin{aligned} 4y(1-y) \frac{d^2}{dy^2} Y + 4[1+2a-(2a+2b+1)y] \frac{d}{dy} Y + \\ + \left[ -4(a+b)^2 - \omega^2 + (4a^2 - m^2) \frac{1}{y} + 4b(b-1) \frac{1}{1-y} \right] Y = 0 \end{aligned} \quad (13.4)$$

Requiring

$$m = \pm 2a, \quad b = 1, \quad b = 0, \quad (13.5)$$

we get an equation of hypergeometric type with

$$\gamma = 1 + 2a, \quad \alpha + \beta = 2a + 2b, \quad \alpha\beta = \frac{4(a+b)^2 + \omega^2}{4},$$

that is

$$\gamma = 1 + 2a, \quad \alpha = a + b \mp \frac{i\omega}{2}, \quad \beta = a + b \pm \frac{i\omega}{2}. \quad (13.6)$$

## 14 Solutions with arbitrary $m, k$ in space $H_3$

Now let us consider radial equations in general case (11.3); the first equation in (11.3) turns to be the identity  $0 = 0$  when three remaining hold:

$$\begin{aligned} -\omega f_1 &= \frac{ik}{\cosh r} f_2 - \frac{im}{\sinh r} f_3, \\ -\omega f_2 &= \left(\frac{d}{dr} + \frac{\sinh r}{\cosh r}\right) f_3 - \frac{ik}{\cosh r} f_1, \\ -\omega f_3 &= -\left(\frac{d}{dr} + \frac{\cosh r}{\sinh r}\right) f_2 + \frac{im}{\sinh r} f_1; \end{aligned}$$

with substitutions

$$f_2 = \frac{1}{\sinh r} F_2, \quad f_3 = \frac{1}{\cosh r} F_3,$$

one obtains

$$\begin{aligned}
-\omega f_1 &= i \frac{k F_2 - m F_3}{\sinh r \cosh r} , \\
-\omega \frac{F_2}{\sinh r} &= \frac{1}{\cosh r} \frac{dF_3}{dr} - \frac{ik}{\cosh r} f_1 , \\
-\omega \frac{F_3}{\cosh r} &= -\frac{1}{\sinh r} \frac{dF_2}{dr} + \frac{im}{\sinh r} f_1 .
\end{aligned} \tag{14.1}$$

After excluding of  $f_1$  we get

$$\begin{aligned}
\left( \frac{\omega}{\cosh r} \frac{d}{dr} + \frac{km}{\sinh r \cosh^2 r} \right) F_3 + \left( \frac{\omega^2}{\sinh r} - \frac{k^2}{\sinh r \cosh^2 r} \right) F_2 &= 0 , \\
\left( \frac{\omega}{\sinh r} \frac{d}{dr} - \frac{km}{\cosh r \sinh^2 r} \right) F_2 - \left( \frac{\omega^2}{\cosh r} - \frac{m^2}{\cosh r \sinh^2 r} \right) F_3 &= 0 .
\end{aligned} \tag{14.2}$$

In variable  $\cosh 2r - 1 = 2y$ , eqs. (14.2) take the form

$$\begin{aligned}
\left( 2\omega \frac{d}{dy} - \frac{km}{y(1+y)} \right) F_2 + \left( -\frac{\omega^2}{1+y} + \frac{m^2}{y(1+y)} \right) F_3 &= 0 , \\
\left( 2\omega \frac{d}{dy} + \frac{km}{y(1+y)} \right) F_3 + \left( +\frac{\omega^2}{y} - \frac{k^2}{y(1+y)} \right) F_2 &= 0 .
\end{aligned} \tag{14.3}$$

Let us introduce new functions by means of a linear transformation (with unit determinant  $\alpha N - \beta M = 1$  and numerical parameters)

$$\begin{aligned}
F_2 &= \alpha G_2 + \beta G_3 , \\
F_3 &= M G_2 + N G_3 ;
\end{aligned} \tag{14.4}$$

and inverse one given by

$$\begin{aligned}
G_2 &= N F_2 - \beta F_3 , \\
G_3 &= -M F_2 + \alpha F_3 .
\end{aligned} \tag{14.5}$$

Combining equations in (14.3), we get

$$\begin{aligned}
2\omega \frac{d}{dy} G_2 - N \frac{km}{y(1+y)} F_2 + N \left( -\frac{\omega^2}{1+y} + \frac{m^2}{y(1+y)} \right) F_3 - \\
-\beta \frac{km}{y(1+y)} F_3 - \beta \left( +\frac{\omega^2}{y} - \frac{k^2}{y(1+y)} \right) F_2 &= 0 , \\
2\omega \frac{d}{dy} G_3 + M \frac{km}{y(1+y)} F_2 - M \left( -\frac{\omega^2}{1+y} + \frac{m^2}{y(1+y)} \right) F_3 + \\
+\alpha \frac{km}{y(1+y)} F_3 + \alpha \left( +\frac{\omega^2}{y} - \frac{k^2}{y(1+y)} \right) F_2 &= 0 .
\end{aligned} \tag{14.6}$$

Expressing  $F_2, F_3$  through  $G_2, G_3$  according to (14.4):

$$2\omega \frac{dG_2}{dy} + \left[ -(N\alpha + \beta M) \frac{km}{y(1+y)} + NM \frac{-\omega^2 y + m^2}{y(1+y)} - \beta\alpha \frac{\omega^2(1+y) - k^2}{y(1+y)} \right] G_2 + \\ + \left[ -2N\beta \frac{km}{y(1+y)} + N^2 \frac{-\omega^2 y + m^2}{y(1+y)} - \beta^2 \frac{\omega^2(1+y) - k^2}{y(1-y)} \right] G_3 = 0, \quad (14.7)$$

$$2\omega \frac{dG_3}{dy} + \left[ (M\beta + \alpha N) \frac{km}{y(1+y)} - NM \frac{-\omega^2 y + m^2}{y(1+y)} + \beta\alpha \frac{\omega^2(1+y) - k^2}{y(1+y)} \right] G_3 + \\ + \left[ 2M\alpha \frac{km}{y(1+y)} - M^2 \frac{-\omega^2 y + m^2}{y(1+y)} + \alpha^2 \frac{\omega^2(1+y) - k^2}{y(1+y)} \right] G_2 = 0, \quad (14.8)$$

Let us detail the coefficients at  $G_3$  and  $G_2$ :

$$\left[ -2N\beta \frac{km}{y(1+y)} + N^2 \frac{-\omega^2 y + m^2}{y(1+y)} - \beta^2 \frac{\omega^2(1+y) - k^2}{y(1-y)} \right] G_3 = \\ = \frac{1}{y(1+y)} [ -2km N\beta - \omega^2(N^2 + \beta^2) y + N^2 m^2 + \beta^2 k^2 ], \\ \left[ 2M\alpha \frac{km}{y(1+y)} - M^2 \frac{-\omega^2 y + m^2}{y(1+y)} + \alpha^2 \frac{\omega^2(1+y) - k^2}{y(1+y)} \right] G_2 = \\ = \frac{1}{y(1+y)} [ 2km M\alpha + \omega^2(M^2 + \alpha^2) y - M^2 m^2 - \alpha^2 k^2 ]$$

Redundant singularities will be excluded if

$$N^2 + \beta^2 = 0, \quad M^2 + \alpha^2 = 0, \quad (14.9)$$

With additional assumption of unitarity for transformations (14.4) and (14.5)

$$\alpha G_2 + \beta G_3 = \cos A G_2 + i \sin A G_3, \\ M G_2 + N G_3 = i \sin A G_2 + \cos A G_3, \quad (14.10)$$

we get

$$N\alpha + \beta M = \cos 2A, \quad 2N\beta = i \sin 2A, \quad 2M\alpha = i \sin 2A, \\ \alpha\beta = i \sin A \cos A = \frac{i}{2} \sin 2A, \quad NM = i \sin A \cos A = \frac{i}{2} \sin 2A, \\ N^2 = \cos^2 A, \quad \beta^2 = -\sin^2 A, \quad M^2 = -\sin^2 A, \quad \alpha^2 = \cos^2 A;$$

correspondingly eqs. (14.7) and (14.8) give

$$2\omega \frac{dG_2}{dy} + \left[ -\cos 2A \frac{km}{y(1+y)} + \frac{i}{2} \sin 2A \frac{-\omega^2 y + m^2 - \omega^2(1+y) + k^2}{y(1+y)} \right] G_2 + \\ + \left[ -i \sin 2A \frac{km}{y(1+y)} + \cos^2 A \frac{-\omega^2 y + m^2}{y(1+y)} + \sin^2 A \frac{\omega^2(1+y) - k^2}{y(1+y)} \right] G_3 = 0,$$

$$2\omega \frac{dG_3}{dy} + \left[ \cos 2A \frac{km}{y(1+y)} - \frac{i}{2} \sin 2A \frac{-\omega^2 y + m^2 - \omega^2(1+y) + k^2}{y(1+y)} \right] G_3 + \\ + \left[ i \sin 2A \frac{km}{y(1+y)} + \sin^2 A \frac{-\omega^2 y + m^2}{y(1+y)} + \cos^2 A \frac{\omega^2(1+y) - k^2}{y(1+y)} \right] G_2 = 0 ,$$

With additional requirement

$$A = \pi/4 , \quad \cos^2 A = \sin^2 A = \frac{1}{2} , \quad \sin 2A = 1 , \quad \cos 2A = 0 ; \quad (14.11)$$

the system becomes much more simple

$$2\omega \frac{dG_2}{dy} + i \frac{-\omega^2 y + m^2 - \omega^2(1+y) + k^2}{2y(1+y)} G_2 + \frac{-2ikm - \omega^2 y + m^2 + \omega^2(1+y) - k^2}{2y(1+y)} G_3 = 0 ,$$

$$2\omega \frac{dG_3}{dy} - i \frac{-\omega^2 y + m^2 - \omega^2(1+y) + k^2}{2y(1+y)} G_3 + \frac{2ikm - \omega^2 y + m^2 + \omega^2(1+y) - k^2}{2y(1+y)} G_2 = 0 ,$$

or

$$\left( 2\omega \frac{d}{dy} - i \frac{\omega^2 y + \omega^2(1+y) - m^2 - k^2}{2y(1+y)} \right) G_2 + \frac{\omega^2 + (m - ik)^2}{2y(1+y)} G_3 = 0 , \\ \left( 2\omega \frac{d}{dy} + i \frac{\omega^2 y + \omega^2(1+y) - m^2 - k^2}{2y(1+y)} \right) G_3 + \frac{\omega^2 + (m + ik)^2}{2y(1+y)} G_2 = 0 . \quad (14.12)$$

From whence it follows that

$$G_3 = -2\omega \frac{2y(1+y)}{\omega^2 + (m - ik)^2} \frac{dG_2}{dy} + i \frac{\omega^2 y + \omega^2(1+y) - m^2 - k^2}{\omega^2 + (m - ik)^2} G_2 , \\ 4y(1+y) \frac{d^2 G_2}{dy^2} + 4(1+2y) \frac{dG_2}{dy} + \left( -2i\omega + \omega^2 - \frac{m^2}{y(1+y)} - \frac{m^2 + k^2}{1+y} \right) G_2 = 0 . \quad (14.13)$$

After changing the variable  $y$  to  $-y$  eq. (14.13) reads

$$4y(1-y) \frac{d^2 G_2}{dy^2} + 4(1-2y) \frac{dG_2}{dy} - \\ - \left( -2i\omega + \omega^2 + \frac{m^2}{y(1-y)} - \frac{m^2 + k^2}{1-y} \right) G_2 = 0 \quad (14.14)$$

Making substitution

$$G_2 = y^A (1-y)^B G(y) ,$$



we get

$$4y(1-y)\frac{d^2G}{dy^2} + 4[1+2A-(2A+2B+1+1)y]\frac{dG}{dy} - \left[ -\omega(2i-\omega) + 4(A+B)(A+B+1) - \frac{4A^2-m^2}{y} - \frac{4B^2+k^2}{1-y} \right] G = 0. \quad (14.15)$$

With additional requirements

$$m = \pm 2A, \quad k = \pm 2iB. \quad (14.16)$$

it becomes an equation of hypergeometric type

$$4y(1-y)\frac{d^2G}{dy^2} + 4[1+2A-(2A+2B+1+1)y]\frac{dG}{dy} - [-\omega(2i-\omega) + 4(A+B)(A+B+1)]G = 0, \quad (14.17)$$

$$\begin{aligned} \gamma &= 1+2A, & \alpha + \beta &= 2A+2B+1, \\ \alpha\beta &= \frac{-\omega(2i-\omega) + 4(A+B)(A+B+1)}{4}, \end{aligned}$$

that is

$$\alpha = A+B - \frac{i\omega}{2}, \quad \beta = A+B+1 + \frac{i\omega}{2}. \quad (14.18)$$

In the same manner, from eqs. (14.12) it follows

$$\begin{aligned} G_2 &= -2\omega \frac{2y(1+y)}{\omega^2 + (m+ik)^2} \frac{dG_3}{dy} - i \frac{\omega^2 y + \omega^2(1+y) - m^2 - k^2}{\omega^2 + (m+ik)^2} G_3, \\ 2\omega \frac{d}{dy} \left( -2\omega \frac{2y(1+y)}{\omega^2 + (m+ik)^2} \frac{dG_3}{dy} - i \frac{\omega^2 y + \omega^2(1+y) - m^2 - k^2}{\omega^2 + (m+ik)^2} G_3 \right) &- \\ -i \frac{\omega^2 y + \omega^2(1+y) - m^2 - k^2}{2y(1+y)} \left( -2\omega \frac{2y(1+y)}{\omega^2 + (m+ik)^2} \frac{dG_3}{dy} - \right. &- \\ \left. -i \frac{\omega^2 y + \omega^2(1+y) - m^2 - k^2}{\omega^2 + (m+ik)^2} G_3 \right) + \frac{\omega^2 + (m-ik)^2}{2y(1+y)} G_3 &= 0, \end{aligned} \quad (14.19)$$

or

$$\begin{aligned} &4y(1+y)\frac{d^2G_3}{dy^2} + 4(1+2y)\frac{dG_3}{dy} + \\ &+ \left( 2i\omega + \omega^2 - \frac{m^2}{y(1+y)} - \frac{m^2+k^2}{1+y} \right) G_3 = 0 \end{aligned} \quad (14.20)$$

After changing the variable  $y$  to  $-y$

$$\begin{aligned} &4y(1-y)\frac{d^2G_3}{dy^2} + 4(1-2y)\frac{dG_3}{dy} - \\ &- \left( 2i\omega + \omega^2 + \frac{m^2}{y(1-y)} - \frac{m^2+k^2}{1-y} \right) G_3 = 0. \end{aligned} \quad (14.21)$$

and with the substitution

$$G_3 = y^A (1-y)^B G(y) ,$$

we get

$$4y(1-y) \frac{d^2 G}{dy^2} + 4[1 + 2A - (2A + 2B + 1 + 1)y] \frac{dG}{dy} - \left[ \omega(2i + \omega) + 4(A + B)(A + B + 1) - \frac{4A^2 - m^2}{y} - \frac{4B^2 + k^2}{1-y} \right] G = 0 . \quad (14.22)$$

With the help of additional restriction

$$m = \pm 2A , \quad k = \pm 2iB . \quad (14.23)$$

the latter reads as an equation of hypergeometric type

$$4y(1-y) \frac{d^2 G}{dy^2} + 4[1 + 2A - (2A + 2B + 1 + 1)y] \frac{dG}{dy} - [\omega(2i + \omega) + 4(A + B)(A + B + 1)] G = 0 , \quad (14.24)$$

$$c = 1 + 2A , \quad a + b = 2A + 2B + 1 , \\ ab = \frac{\omega(2i + \omega) + 4(A + B)(A + B + 1)}{4} ,$$

that is

$$a = A + B + 1 - \frac{i\omega}{2} , \quad b = A + B + \frac{i\omega}{2} . \quad (14.25)$$

Let us find a relative factor in two functions  $G_2$  and  $G_3$ . Starting from

$$G_2 = M_2 y^{|m|/2} (1-y)^{|k|/2i} F(-n, n+1+|m|+\frac{|k|}{i}, |m|+1; y) = \\ = M_2 y^{(c-1)/2} (1-y)^{(a+b-c)/2} F(a, b, c, y) . \quad (14.26)$$

Then

$$G_3 = M_3 y^{|m|/2i} (1-y)^{|k|/2i} F(-n+1, n+|m|+\frac{|k|}{i}, |m|+1; y) = \\ = M_3 y^{(c-1)/2} (1-y)^{(a+b-c)/2} F(a+1, b-1, c, y) . \quad (14.27)$$

and allowing for

$$G_3 [(m - ik)^2 + \omega^2] = -4\omega y(1-y) \frac{dG_2}{dy} + i[-m^2 - k^2 + \omega^2(1-2y)] G_2 . \quad (14.28)$$

we arrive at

$$\begin{aligned}
& (m - ik - i\omega) (m - ik + i\omega) \frac{M_3}{M_2} F_3(y) = \\
& = -4\omega \left[ \frac{|m|}{2} (1 - y) F_2(y) - \frac{|k|}{2i} y F_2(y) + \right. \\
& \left. + y(1 - y) \frac{d}{dy} F_2(y) \right] + i[-m^2 - k^2 + \omega^2(1 - 2y)] F_2(y) .
\end{aligned} \tag{14.29}$$

To find the relative factor it sufficient to consider the latter at the point  $y = 0$  which results in

$$(m - ik - i\omega) (m - ik + i\omega) \frac{M_3}{M_2} = -2\omega |m| - im^2 - ik^2 + i\omega^2 .$$

or

$$i(-i\omega + m - ik) (i\omega + m - ik) \frac{M_3}{M_2} = (-i\omega + |m| - ik) (i\omega + |m| + ik) ,$$

that is

$$\begin{aligned}
M_2 &= iM (-i\omega + m - ik)(i\omega + m - ik) , \\
M_3 &= M (-i\omega + |m| - ik)(i\omega + |m| + ik) .
\end{aligned} \tag{14.30}$$

These relations become more simple when separating regions for  $m$ :

$$\begin{aligned}
m > 0 , \quad M_2 &= iM(i\omega - ik + m) , \quad M_3 = M(i\omega + ik + m) ; \\
m < 0 , \quad M_2 &= iM(i\omega - ik + m) , \quad M_3 = M(i\omega + ik + m) ; \\
m = 0 , \quad M_2 &= M(k - \omega) , \quad M_3 = iM(k + \omega) .
\end{aligned}$$

## 15 Discussion: on relation to other formalisms in Maxwell theory

There exist close relation between the above used covariant technique in the complex form of Riemann-Silberstein-Majorana-Oppenheimer and spinor form of Maxwell theory in general relativity [18] mainly used in the form of Newman-Penrose formalism of isotropic tetrad [25]:

$$\begin{aligned}
i\sigma^\alpha(x) [ \partial/\partial x^\alpha + \Sigma_\alpha(x) \otimes I + I \otimes \Sigma_\alpha(x) ] \xi(x) &= -j(x) , \\
i\bar{\sigma}^\alpha(x) [ \partial/\partial x^\alpha + \bar{\Sigma}_\alpha(x) \otimes I + I \otimes \bar{\Sigma}_\alpha(x) ] \eta(x) &= -\bar{j}(x) , \\
\nabla^\beta F_{\alpha\beta}(x) &= -j_\alpha(x) , \quad \epsilon^{\alpha\beta\rho\sigma} \nabla_\beta F_{\rho\sigma}(x) = 0 ;
\end{aligned} \tag{15.1}$$

$$\begin{aligned}
i\bar{\sigma}^\alpha(x) [ \partial/\partial x^\alpha + \bar{\Sigma}_\alpha(x) \otimes I + I \otimes \Sigma_\alpha(x) ] H(x) &= \xi , \\
i\sigma^\alpha(x) [ \partial/\partial x^\alpha + \Sigma_\alpha(x) \otimes I + I \otimes \bar{\Sigma}_\alpha(x) ] \Delta(x) &= \eta , \\
\nabla^\alpha A_\alpha &= 0 , \quad \nabla_\alpha A_\beta - \nabla_\beta A_\alpha = F_{\alpha\beta} ;
\end{aligned} \tag{15.2}$$

where spinor connections by Infeld – van der Vaerden are used

$$\begin{aligned}\sigma^\alpha(x) &= \sigma^a e_{(a)}^\alpha(x), & \bar{\sigma}^\alpha(x) &= \bar{\sigma}^a e_{(a)}^\alpha(x), \\ \Sigma_\alpha(x) &= \frac{1}{2} \Sigma^{ab} e_{(a)}^\beta \nabla_\alpha(e_{(b)\beta}), & \bar{\Sigma}_\alpha(x) &= \frac{1}{2} \bar{\Sigma}^{ab} e_{(a)}^\beta \nabla_\alpha(e_{(b)\beta}), \\ \Sigma^a &= \frac{1}{4} (\bar{\sigma}^a \sigma^b - \bar{\sigma}^b \sigma^a), & \bar{\Sigma}^a &= \frac{1}{4} (\sigma^a \bar{\sigma}^b - \sigma^b \bar{\sigma}^a); \end{aligned} \quad (15.3)$$

and electromagnetic bi-spinor and electric source are given by

$$\begin{aligned} \left| \begin{array}{cc} \xi & \Delta \\ H & \eta \end{array} \right| &= \left| \begin{array}{cc} +\Sigma^{mn} \sigma^2 F_{mn} & +i \bar{\sigma}^l \sigma^2 A_l \\ -i \sigma^l \sigma^2 A_l & -\bar{\Sigma}^{mn} \sigma^2 F_{mn} \end{array} \right|; \\ j(x) &= i \sigma^k \sigma^2 j_k(x), & \bar{j}(x) &= -i \bar{\sigma}^k \sigma^2 j_k(x). \end{aligned} \quad (15.4)$$

Used in the paper complex 3-vector approach seems to be a good alternative to the other possible techniques. Let us summarize the content of the paper again.

Complex formalism of Riemann - Silberstein - Majorana - Oppenheimer in Maxwell electrodynamics is extended to the case of arbitrary pseudo-Riemannian space - time in accordance with the tetrad recipe of Tetrode - Weyl - Fock - Ivanenko. In this approach, the Maxwell equations are solved exactly on the background of static cosmological Einstein model, parameterized by special cylindrical coordinates and realized as a Riemann space of constant positive curvature. A discrete frequency spectrum for electromagnetic modes depending on the curvature radius of space and three parameters is found, and corresponding basis electromagnetic solutions have been constructed explicitly. In the case of elliptical model a part of the constructed solutions should be rejected by continuity considerations.

Similar treatment is given for Maxwell equations in hyperbolic Lobachevsky model, the complete basis of electromagnetic solutions in corresponding cylindrical coordinates has been constructed as well, no quantization of frequencies of electromagnetic modes arises.

## 16 Supplement: on the use of matrix complex form of the Maxwell equations construct electromagnetic solutions from scalar ones in Minkowski space-time

The above matrix form of Maxwell theory :

$$(-i\partial_0 + \alpha^j \partial_j) \Psi = 0, \quad \Psi = \begin{vmatrix} 0 \\ \psi^1 \\ \psi^2 \\ \psi^3 \end{vmatrix}. \quad (16.1)$$

being applied to the flat Minkowski model, permits us to develop a simple method of finding solutions of Maxwell equations on the base of known solutions of the scalar massless equation by Klein – Fock – Gordon. Indeed, in virtue of the above commutative relations we have an operator identity

$$(-i\partial_0 + \alpha^1 \partial_1 + \alpha^2 \partial_2 + \alpha^3 \partial_3) (-i\partial_0 - \alpha^1 \partial_1 - \alpha^2 \partial_2 - \alpha^3 \partial_3) = (-\partial_0^2 + \partial_1^2 + \partial_2^2 + \partial_3^2).$$

Therefore, taking any special scalar solution one can immediately construct four solutions of the Maxwell equation:

$$(i\partial_0 + \alpha^1\partial_1 + \alpha^2\partial_2 + \alpha^3\partial_3) \Phi(x) =$$

$$= \begin{vmatrix} i\partial_0\Phi & \partial_1\Phi & \partial_2\Phi & \partial_3\Phi \\ -\partial_1\Phi & i\partial_0\Phi & -\partial_3\Phi & \partial_2\Phi \\ -\partial_2\Phi & \partial_3\Phi & i\partial_0\Phi & -\partial_1\Phi \\ -\partial_3\Phi & -\partial_2\Phi & \partial_1\Phi & i\partial_0\Phi \end{vmatrix} = \{\Psi^0, \Psi^1, \Psi^2, \Psi^3\}. \quad (16.2)$$

Thus, we have four formal solutions of the free Maxwell equations (let  $F_a(x) = \partial_a\Phi(x)$ ):

$$\{\Psi^0, \Psi^1, \Psi^2, \Psi^3\} = \begin{vmatrix} iF_0 & F_1 & F_2 & F_3 \\ -F_1 & iF_0 & -F_3 & F_2 \\ -F_2 & F_3 & iF_0 & -F_1 \\ -F_3 & -F_2 & F_1 & iF_0 \end{vmatrix}. \quad (16.3)$$

In general, the function  $\Phi(x)$  is a complex-valued. Relationship defining all possible solutions has the structure

$$\lambda_0\Psi^0 + \lambda_1\Psi^1 + \lambda_2\Psi^2 + \lambda_3\Psi^3 = \begin{vmatrix} 0 \\ \mathbf{E} + ic\mathbf{B} \end{vmatrix}. \quad (16.4)$$

Substituting these expression into Maxwell equations and performing calculations needed we can derive four relations (for more details see [23]):

$$[-\lambda_0\partial_0 + i\lambda_j\partial_j]F_c = 0. \quad (16.5)$$

These equations should be analyzed instead of Maxwell equations to construct solutions of them in any coordinate system.

It seems reasonable to expect further developments in this matrix based approach to Maxwell theory, as a possible base to explore general method to separate the variables for Maxwell equations in different coordinates (see [24]).

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